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FOUNDATIONS

Wang, Hao. The irreducibility of impredicative principles. Math. Ann. 125, 56–66 (1952).

By an impredicative set we mean a set defined by referring to a totality to which the set itself belongs. This paper shows that in general it is not possible to reduce the ways of generating impredicative sets to finitely many. The proof is carried out for Zermelo set theory with the axiom of separation. Let this system be Z. It is shown that Z cannot be replaced by any finite set of provable sentences of the system. The proof is by contradiction: suppose S is a finite set of sentences of Z from which we can derive all of Z. If Z is consistent, so is S and S must then have a denumerable model. Since S is finite, we can enumerate in Z the definable sets of S. By a diagonal procedure, we can then find a new set whose existence can be proved only in Z but not in S. Hence Z is not finitely axiomatisable. Some of the details of the proof are not given in this paper. There is also a discussion of similar results in other systems.

I. Novak Gál (Ithaca, N. Y.).

Wang, Hao. Arithmetic models for formal systems. Methodos 3, 217–232 (1951).

By saying that a formal system S has a model in a formal system S' we mean, generally speaking, that a certain effective translation of S into S' carries over either theorems of S into theorems of S' (stronger sense) or theorems of S into true sentences of S' (weaker sense). If for S we take the Zermelo set theory and for S' the system Z of ordinary number theory, then it follows from Gödel's incompleteness theorem that S has a model in Z in the weaker but not in the stronger sense, provided S is consistent. More precisely, the following main theorem is proved: If Con(S) is the arithmetization of a statement expressing the consistency of S, then S has a model in the weaker sense in the system Z_S resulting from Z by adding Con(S) as an axiom (this remains true if S is inconsistent; for in this case Con(S) can be refuted in Z, hence Z_S is also inconsistent). This theorem is proved by arithmetizing the proof of the Löwenheim-Skolem-Gödel theorem. In a final section, the author discusses the notion of the modelling of an axiom system which was applied in the first part of the paper. *E. W. Beth.*

Kleene, S. C. Permutability of inferences in Gentzen's calculi *LK* and *LJ*. Two papers on the predicate calculus. Mem. Amer. Math. Soc., no. 10, pp. 1–26 (1952). $1.30.

Die Erweiterung des Gentzenschen Hauptsatzes besagt, dass es im (klassischen) Sequenzenkalkül zu einer ableitbaren Sequenz aus pränexen Formeln allemal eine Ableitung gibt, in der alle aussagenlogischen Regeln allen prädikatenlogischen Regeln vorangehen. Verf. gibt Bedingungen an— für den klassischen und den intuitionistischen Sequenzenkalkül—, denen eine Klasseneinteilung der logischen Konstanten (die an Stelle der Einteilung in aussagenlogische und prädikatenlogische Konstanten tritt) genügen muss, damit ein entsprechender Satz gilt. *P. Lorenzen.*

Kleene, S. C. Finite axiomatizability of theories in the predicate calculus using additional predicate symbols. Two papers on the predicate calculus. Mem. Amer. Math. Soc., no. 10, pp. 27–68 (1952). $1.30.

Eine Klasse $\mathfrak{C}$ von P-Formeln (Formeln des engeren Prädikatenkalküls mit Prädikatensymbolen aus einem System $P_1, \cdots, P_n$), die (1) rekursiv aufzählbar, (2) abgeschlossen bzgl. der Ableitbarkeit im Prädikatenkalkül ist, ist "axiomatisierbar", d.h. es gibt eine rekursive Klasse $\mathfrak{A}$ von P-Formeln, so dass (3) $\mathfrak{C}$ die abgeschlossene Hülle von $\mathfrak{A}$ ist. Die formalisierte Mengenlehre und die formalisierte Arithmetik zeigen, dass $\mathfrak{C}$ nicht allemal "endlich axiomatisierbar ist", d.h. es existiert manchmal keine endliche Klasse $\mathfrak{A}$ mit (3). [Vgl. H. Wang, Proc. Nat. Acad. Sci. U. S. A. 36, 479–484 (1950); diese Rev. 12, 469] Verf. beweist, dass jede solche Klasse $\mathfrak{C}$ aber "endlich axiomatisierbar mit zusätzlichen Prädikatensymbolen" ist, d.h. es gibt eine endlich Klasse $\mathfrak{A}$ von Formeln, die ausser $P_1, \cdots, P_n$ noch andere Prädikatensymbole enthalten, derart, dass eine P-Formel $\mathfrak{A}$ genau dann in $\mathfrak{C}$ liegt, wenn A aus $\mathfrak{A}$ ableitbar ist. Dem Beweis liegt—in äusserster Vereinfachung—der folgende Gedankengang zugrunde. Wegen (1) lässt sich jeder P-formel A eine Zahl $H(A)$ so zuordnen, dass es eine primitiv-rekursive Funktion ϕ gibt mit

(4) $H(A)=h$ für ein $A \varepsilon \mathfrak{C} \leftrightarrow \phi(s)=h$ für eine Zahl s.

Es lässt sich daher ein Formalismus S_1—ohne logische Konstanten—aufstellen mit nur endlichvielen Funktionssymbolen $', \cdots, f$ und dem Individuensymbol 0, für den gilt

(5) $$H(A)=h \text{ für ein } A \varepsilon \mathfrak{C} \leftrightarrow (\exists s) f(s)=\underline{h}$$

ist in S_1 ableitbar wenn $\underline{h}$ der Ausdruck $0''\cdots'$ mit h Strichen ist. S_1 wird erweitert zu einem Formalismus S, der den Prädikatenkalkül enthält und ausser $P_1, \cdots, P_n$ ein weiteres Prädikatensymbol N mit dem Axiom

$$f(s)=y \supset N(y)$$

und endlichvielen weiteren Axiomen, die für jede P-Formel A mit $h=H(A)$ abzuleiten gestatten:

$$N(\underline{h}) \supset A \quad \text{und} \quad A \supset N(\underline{h}).$$

Es gilt dann für jede P-formel A:

(6) $A \varepsilon \mathfrak{C} \rightarrow A$ ist in S ableitbar.

Mit nicht-finiten Mitteln ist wegen (2) leicht die Umkehrung von (6) zu gewinnen. Verf. führt aber einen metamathematischen Beweis mit finiten Mittlen, der für den klassischen und den intuitionistischen Prädikatenkalkül gilt. Der Beweis kompliziert sich dadurch, dass man sich nicht auf Formeln ohne freie Variable—wie bei der obigen Skizze —beschränken kann. Wesentliches Beweismittel ist der Sequenzenkalkül mit dem Gentzenschen Hauptsatz und seinen Erweiterungen [vgl. das vorstehende Referat].

P. Lorenzen (Bonn).

Goodman, Nelson. New notes on simplicity. J. Symbolic Logic **17**, 189–191 (1952).

This paper contains a number of amendments to earlier papers [same J. **8**, 107–121 (1943); **14**, 32–41 (1949); these Rev. **5**, 86; **10**, 668] and to the author's recent book on "The structure of appearance" [Harvard Univ. Press, 1951]. In particular, it is shown that every predicate can be replaced by irreflexive predicates; hence we can require every primitive predicate to be replaced by the simplest adequate set of irreflexive predicates before the complexity value of a basis is calculated. Under the revised method of computation, the maximum complexity value of an n-place predicate becomes $(2^n-1)(n-1)+1$. *E. W. Beth* (Amsterdam).

Myhill, John. The hypothesis that all classes are nameable. Proc. Nat. Acad. Sci. U. S. A. **38**, 979–981 (1952).

Ist das v. Neumann-Bernayssche Axiomensystem der Mengenlehre widerspruchsfrei (wf.), dann ist nach Gödel auch die Hinzufügung des Axioms $\theta=\mathfrak{L}$, das ausdrückt, dass jede Menge "constructible" ist, wf. Verf. fügt ein weiteres Axiom hinzu, das ausdrückt, dass jede Klasse "nameable" ist. Auch hierdurch wird an der evtl. Wf. nichts geändert.

P. Lorenzen (Bonn).

Moh, Shaw-Kwei. A note on the theory of quantification. J. Symbolic Logic **17**, 243–244 (1952).

Quine [same J. **10**, 1–12 (1945); these Rev. **7**, 45] formulated a convenient decision procedure for the monadic functional calculus of first order and showed that polyadic theory can be obtained by adding a generalized modus ponens. Using a result of Berry [ibid. **6**, 23–27 (1941); these Rev. **2**, 209], the author shows that a weakened modus ponens is sufficient. *R. Barcan Marcus* (Evanston, Ill.).

Rosser, J. Barkley. The axiom of infinity in Quine's New Foundations. J. Symbolic Logic **17**, 238–242 (1952).

Results are obtained concerning Quine's New Foundations [Amer. Math. Monthly **44**, 70–80 (1937)], the latter with axiom of infinity, the New Foundations with classes as proposed by Wang [J. Symbolic Logic **15**, 25–32 (1950); these Rev. **11**, 636], and the system of Quine's "Mathematical logic" [revised ed., Harvard Univ. Press, 1951; these Rev. **13**, 613]. These are called NF, $NF+AF$, $NF+C$, and ML respectively. The conclusions are: (1) In NF the axiom of infinity $\sim(\Lambda \varepsilon Nn)$ is equivalent to the definability of an ordered pair of type zero. (2) The axiom of infinity is not necessarily provable in NF although the following hold: a formula of NF which is provable in NF is provable in $NF+C$ and conversely; there is an essential identity between $NF+C$ and ML; the axiom of infinity is provable in ML. The given proof of $\sim(\Lambda \varepsilon Nn)$ in ML is adequate only to prove the weaker $\sim(\Lambda \varepsilon Nn(C))$ in NF where "$Nn(C)$" is the expression in $NF+C$ which corresponds to "Nn" in ML. It would seem therefore that there is a respect in which ML is weaker than $NF+AF$. (3) To go beyond elementary calculus in $NF+C$ and hence ML requires elementhood for $Nn(C)$. It is shown (using a lemma communicated by Wang) that the latter cannot be proved in $NF+C$ unless NF (and consequently $NF+C$ and ML) is inconsistent. (4) Although NF has no standard model [Rosser and Wang, J. Symbolic Logic **15**, 113–129 (1950); these Rev. **12**, 384], this does not preclude a standard model for $NF+C$ and ML; nor would ω-inconsistency of NF preclude ω-inconsistency of $NF+C$ and ML. *R. Barcan Marcus* (Evanston, Ill.).

Quine, W. V. The problem of simplifying truth functions. Amer. Math. Monthly **59**, 521–531 (1952).

Jede Formel der Aussagenlogik kann auf eine disjunktive Normalform (*) $s_{11}s_{12}\cdots \vee s_{21}s_{22}\cdots \vee \cdots$ gebracht werden, in der $s_{\mu\nu}$ eine Aussagenvariable oder Negation einer solchen ist. Ist keine Formel, die aus (*) durch Weglassen eines $s_{\mu 1}s_{\mu 2}\cdots$ oder eines $s_{\mu\nu}$ entsteht, äquivalent zu (*), dann heisst (*) irredundant. Zu manchen irredundanten Formeln gibt es "einfachere" äquivalente Formeln, z.B., $p\bar{q} \vee \bar{p}r \vee q\bar{r}$ zu $p\bar{q} \vee \bar{p}q \vee q\bar{r} \vee \bar{q}r$. Verf. entwickelt eine Methode, zu jeder Formel alle äquivalenten irredundanten Formeln aufzustellen. Die Methode ist zur Vereinfachung elektrischer Schaltungen brauchbar. *P. Lorenzen* (Bonn).

Swift, J. D. Algebraic properties of N-valued propositional calculi. Amer. Math. Monthly **59**, 612–621 (1952).

In this paper n-valued propositional calculi are investigated from several algebraic standpoints. After considering the general algebra in terms of its basic structure the author discusses the problem of determining under what operations the binary composition algebra, i.e., the algebra of all functions defined on the range $0, 1, \cdots, n-1$ to the range, is a ring. He then considers partial orderings of the range and discusses briefly the lattice properties of the algebra and of the ideals in the ring structure. Finally, in the 3-valued case, all commutative Sheffer functions are listed. Except in the last part of the paper some of the material is expository.

A. Rose (Nottingham).

Kreisel, G. On the interpretation of non-finitist proofs. II. Interpretation of number theory. Applications. J. Symbolic Logic **17**, 43–58 (1952).

The author continues the work of part I [same J. **16**, 241–267 (1951); these Rev. **14**, 122] giving a detailed demonstration of the "no-counter-example" interpretation for extensions of the number-theoretic system Z of Hilbert-Bernays obtained by adding free function variables and verifiable free variable formulas as axioms. The complicated work depends upon the ϵ-substitution method of Hilbert-Bernays and Ackermann's consistency proof [see Ackermann, Math. Ann. **117**, 162–194 (1940); these Rev. **1**, 322]. The author applies a method suggested by the general theory to a discussion of Littlewood's theorem that $\pi(n)-\mathrm{li}(n)$ changes sign infinitely often ($\pi(n)$ is the number of primes not exceeding n and $\mathrm{li}(n)$ is the logarithmic integral). Contrary to earlier opinion that the theorem was nonconstructive or that "new ideas" of proof would be required to establish it in a constructive way, the author shows that it admits of a constructive interpretation. He also presents a free-variable calculus conjectured to be adequate for the provability of the free-variable formulas of his interpretations. Erratum supplied by the author: p. 57 read "$a_n=\sum_{m=1}^{\infty}a(n, m)2^{-m}$" instead of "$a_n=\sum_{m=1}^{n}a(n, m)2^{-m}$".

D. Nelson (Washington, D. C.).

***Witt, E. Matemática intuicionista. [Intuitionistic mathematics.]** Conferencias de Matemática, no. II. Instituto de Matemáticas "Jorge Juan," Madrid, 1951. 8 pp.

The author points out that certain theorems of classical mathematics are intuitionistically invalid. The method, that of Brouwer, is based on the assumption that at any given time there can be found an arithmetic predicate $A(n)$ such that the statement "For all n, $A(n)$." is not known at that time to be either true or false. No mention is made of more recent attempts at precise characterization of constructive decidability. The non-validity of the particular theorems

mentioned by the author follows easily from the fact that the law of trichotomy for real numbers is not intuitionistically valid. *D. Nelson* (Washington, D. C.).

✦Dijkman, Jacobus Gerhardus. Convergentie en divergentie in de intuitionistische wiskunde. [Convergence and divergence in intuitionistic mathematics.] Thesis, University of Amsterdam, 's-Gravenhage, 1952. x+98 pp. (Dutch. English summary)

A broad investigation of the notions of intuitionistic convergence and divergence. The definitions are in some respects more systematic than those of Brouwer and Bellinfante, and elementary convergence properties are proved. Manifold and ω-fold negative convergence, monotonic sequences, divergence notions, Cauchy conditions, boundedness, continued fractions, Abel and Tauber theorems, infinite products, and sequences of functions are treated. Logical notations are applied. It is impossible to give briefly a more detailed account of the rich contents.

H. Freudenthal (Utrecht).

van Rootselaar, B. Un problème de M. Dijkman. Nederl. Akad. Wetensch. Proc. Ser. A. **55** = Indagationes Math. **14**, 405–407 (1952).

The author demonstrates the intuitionistic equivalence of two alternative definitions of derivative of a function defined on the unit interval. Using appropriate intuitionistic definitions of limit, he shows that $\lim_{y\to x}[f(y)-f(x)]/(y-x)$ exists if $\lim_{n\to\infty}[f(x_n)-f(x)]/(x_n-x)$ exists and is the same for every sequence of points $\{x_n\}$ of the unit interval which is convergent to x. *D. Nelson* (Washington, D. C.).

Brouwer, L. E. J. On accumulation cores of infinite species of point cores. Nederl. Akad. Wetensch. Proc. Ser. A. **55** = Indagationes Math. **14**, 439–441 (1952). (Dutch)

The author refutes by counterexamples the two (classically equivalent) formulations of the Bolzano-Weierstrass theorem for the straight line: (A) Every bounded infinite species of point cores admits an accumulation core; (B) every bounded species of point cores without accumulation core is numerically bounded. (A species S is numerically bounded if a natural number n can be found such that S contains no subspecies of n elements.) The following weaker statements are proved to be intuitionistically valid: (A′) Let Q be a bounded infinite species of point cores, ϵ any positive number, m any natural number; then an interval of length less than ϵ can be found which contains (at least) m elements of Q; (B′) a species Q of point cores such that for every element x of Q an open interval (a, b), containing x, can be found which cannot contain two different elements of Q is numerically bounded. *A. Heyting* (Amsterdam).

✦Wilder, Raymond L. Introduction to the foundations of mathematics. John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London, 1952. xiv+305 pp. $5.75.

The first part of this book gives, after two introductory chapters dealing with the axiomatic method in general, an exposition of the basic theories of modern mathematics: the theory of sets, the real number system (on the basis of the Peano axioms) and the theory of groups (including some of its applications in algebra and geometry). Special attention is given to those topics which are important from the point of view of research on foundations, such as the relations between various definitions of infinity, diagonal procedures, well-ordering, the choice axiom and its equivalents. The problems added to each chapter contain much interesting material.

The second part is devoted to a discussion on various viewpoints on foundations. After a summary of the earlier developments (up to the Zermelo system), the Frege-Russell thesis, intuitionism, and formalism are more fully explained. A final chapter deals with the cultural setting of mathematics.

The method of treatment seems to be particularly well adapted to the needs of the general mathematician looking for information on foundations. Those who wish to specialize in foundations will need a more systematic treatment of formal methods, but they will find the study of Wilder's book both useful and stimulating. *E. W. Beth.*

✦Katětov, Miroslav. Jaká je logická výstavba matematiky? [What is the logical structure of mathematics?] Jednota Československých Matematiků a Fysiků, Prague, 1950. 98 pp. Kčs 26.00.

An elementary introduction into mathematical logic which does not demand any knowledge of mathematics beyond the secondary school curriculum. In the introduction the author states his negative attitude to the "idealistic" viewpoint. In his opinion mathematical axioms are to be regarded as abstractions and generalizations of relationships in the real world. *J. K. Mayer* (New York, N. Y.).

Knaster, Bronisław. On applications of mathematical logic to mathematics. Časopis Pěst. Mat. **76**, 3–22 (1951). (Czech)

A summary by the author of a series of lectures on mathematical logic and its application to different mathematical problems. Main topics: propositional calculus, quantificators, theory of sets and relations, axiomatization and formalization of mathematical theories and metamathematics. The author concludes his lectures with the statement that logic is as little a priori as geometry or mechanics. According to him this need not lead to an acceptance of relativism and is in no way in contradiction to the point of view of scientific materialsim, since according to Engels "materialism must change its form with every epoch-making discovery." *J. K. Mayer* (New York, N. Y.).

✦Stabler, E. R. An introduction to mathematical thought. Addison-Wesley Publishing Company, Inc., Cambridge, Mass., 1953. xix+268 pp. $4.50.

This book, which intends "to provide a unified and substantial approach to the logical structure of mathematics, and to develop a corresponding philosophical point of view towards mathematical knowledge", brings together the elementary aspects of symbolic logic (pp. 1–122) and of modern algebra (pp. 123–228). It closes with a brief discussion of the foundations of mathematics (pp. 229–255) from the point of view of the postulationists, the logicalists, formalists and intuitionists. One may well question the value of such an "exploratory" treatment for readers whose training in mathematics has not gone beyond "at least plane geometry and intermediate algebra in high school". It provides pleasant reading for students of greater mathematical maturity. One of the indirect values of an introductory treatment of this character is that it is likely to lead thoughtful students to questions, particularly as to the philosophical basis of mathematics, which might otherwise never occur to them. *A. Dresden* (Swarthmore, Pa.).

ALGEBRA

Gruder, Osias. Zur Theorie der Zerlegung von Permutationen in Zyklen. Ark. Mat. 2, 385–414 (1952).

The author first proves the following result: Let $F(n)$ be the number of permutations of degree n which are a product of disjoint cycles whose orders are taken from a given set $\{a_1, a_2, \cdots\}$ of integers. Let $f(k, n)$ be the number of such permutations consisting of exactly k cycles. Put $h(z)=z^{a_1}/a_1+z^{a_2}/a_2+\cdots$. Then

$$e^{h(z)}=\sum F(n)\frac{z^n}{n!}, \quad e^{xh(z)}=\sum_k \sum_n f(k, n)\frac{x^k z^n}{n!}.$$

Applying this result for special choices of the set $(a_1, a_2, \cdots)$, the author obtains without difficulty several theorems due to Sylvester on the number of permutation consisting of: k disjoint cycles; disjoint cycles of odd order; disjoint cycles of even order.

For the set $(a, a+b, a+2b, \cdots)$ the author derives, independently of the previous result, recursion formulas for $f(n, k)$ and $F(n)$. From these he obtains closed expressions for these quantities. Applying this result he obtains two further results of Sylvester on the number of permutations which are a product of k disjoint cycles of odd order and the number of permutations which are a product of k cycles whose order is $\equiv 1$ (m). Generalizing the problème de rencontres and related problems the author develops recursion formulas for the number of permutations whose disjoint cycles have orders $\geqq a$ and for those whose disjoint cycles have orders $<a$. Finally these results are applied to give combinatorial definitions for the numbers e and π and for the Euler constant. *H. B. Mann* (Columbus, Ohio).

Yamamoto, Koichi. On the asymptotic number of Latin rectangles. Jap. J. Math. 21 (1951), 113–119 (1952).

Erdös and Kaplansky [Amer. J. Math. 68, 230–236 (1946); these Rev. 7, 407] proved that the number $f(n, k)$ of n by k Latin rectangles is for $k<(\log n)^{3/2-\delta}$ asymptotically equal to $(n!)^k \exp(-k(k-1)/2)$. The author extends this result to $k<n^{1/3-\delta}$ where δ is arbitrarily small. The proof utilizes the formula of Erdös and Kaplansky for the number of ways in which an n by k Latin rectangle can be augmented to an n by $(k+1)$ Latin rectangle.

H. B. Mann (Columbus, Ohio).

Bose, R. C., and Bush, K. A. Orthogonal arrays of strength two and three. Ann. Math. Statistics 23, 508–524 (1952).

Plackett and Burman [Biometrika 33, 305–325 (1946); these Rev. 8, 44] proved for orthogonal arrays $(\lambda s^2, k, s, 2)$ the inequality $k\leqq[(\lambda s^2-1)/(s-1)]$. R. Rao [Suppl. J. Royal Statist. Soc. 9, 128–139 (1947); these Rev. 9, 264] generalized this result and proved inequalities which imply for an orthogonal array $(\lambda s^3, k, s, 3)$ the inequality $k\leqq[(\lambda s^2-1)/(s-1)]+1$. The authors give new proofs of both inequalities and improve them under the condition that $(\lambda-1)\not\equiv 0\ (s-1)$ and for $(\lambda-1)\equiv 0\ (s-1)$ under certain other conditions on s and λ. A general theorem is then proved for the construction of arrays $(\lambda s^2, k, s, 2)$ from which a method for constructing all arrays of this type is derived for $\lambda=p^u$, $s=p^v$, $k=\lambda(s^{c+1}-1)/(s^c-s^{c-1})+1$, p a prime, $c=[u/v]$. Another general theorem leads to methods for constructing $(s^3, s+2, s, 3)$ if $s=2^n$ and $(s^3, s+1, s, 3)$ if $s=p^n$ where p is an odd prime. According to a recent result by K. A. Bush [Ann. Math. Statist. 23, 426–434 (1952); these Rev. 14, 125] the number of costraints in the last two designs cannot be increased. *H. B. Mann.*

Connor, W. S. Some relations among the blocks of symmetrical group divisible designs. Ann. Math. Statistics 23, 602–609 (1952).

An incomplete block design is called group divisible (G.D.) if the treatments can be divided into m groups of n each such that any two treatments belonging to the same group occur together in λ_1 blocks and any two treatments belonging to two different groups occur together in λ_2 blocks $(\lambda_1\neq\lambda_2)$. The G.D. designs fall into three classes. Singular G.D. designs $r=\lambda_1$, semi-regular G.D. designs $r>\lambda_1$, $rk=\lambda_2 v$ and regular G.D. designs $r>\lambda_1$, $rk>\lambda_2 v$. The author defines incidence matrices and structural matrices in analogy to the same concepts for balanced incomplete block designs [Connor, same Ann. 23, 57–71 (1952); these Rev. 13, 617] and also a new type of structural matrix specific for G.D. designs. The author proves several relations for these structural matrices by means of which he succeeds in showing the following two theorems. (1) For a regular G.D. design $\lambda_2(r-\lambda_1)(r^2-v\lambda_2)\leqq\lambda_1$ if $\lambda_1>\lambda_2$. The reversed inequality holds if $\lambda_1<\lambda_2$. (2) If in a regular G.D. design $r^2-v\lambda_2$ and $\lambda_1-\lambda_2$ are relatively prime, then the blocks fall into m groups of n blocks each which are such that any two blocks from the same group have λ_1 treatments in common and any two blocks from different groups have λ_2 treatments in common. (3) For a semi-regular G.D. design $\lambda_2>\lambda_1$. The conclusion of theorem 2 can sometimes also be reached even if $r^2-v\lambda_2$ and $\lambda_1-\lambda_2$ do have a common factor. The author demonstrates this on the example $v=b=45$, $r=k=9$, $m=3$, $n=15$, $\lambda_1=3$, $\lambda_2=1$. *H. B. Mann* (Columbus, Ohio).

Errera, A. Sur une suite sans répétitions. Mathesis 61, 169–173 (1952).

It is proved that the infinite sequence constructed by Euwe [Nederl. Akad. Wetensch., Proc. 32, 633–642 (1929)], each term of which is one of two elements, to have no three consecutive like subsequences of any length, has in fact the desired property. A derived sequence having three possible elements for each term is shown to have no two consecutive like subsequences, thus disproving a conjecture by Rivier.

J. Riordan (New York, N. Y.).

⋆Krull, Wolfgang. Elementare und klassische Algebra vom modernen Standpunkt. 2d ed. Sammlung Göschen Band 930. Walter de Gruyter & Co., 1952. 136 pp. DM 2.40.

The purpose of this little book is to bridge the gap between classical algebra and modern or abstract algebra, with emphasis on the theory of polynomial equations, and to prepare the student for further work in nonlinear algebra. The author's reluctance to use group theory not only limits the material but also imposes a certain artificiality on some of the theorems and proofs. *E. R. Kolchin.*

⋆Stoll, Robert R. Linear algebra and matrix theory. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1952. xvi+272 pp.

This is an elementary text. The titles of the chapters are: (1) Systems of linear equations; (2) Vector spaces; (3) Basic operations for matrices; (4) Determinants; (5) Bilinear and quadratic functions and forms; (6) Linear transformations on a vector space; (7) Canonical representations of a linear transformation; (8) Unitary and Euclidean vector spaces.

Chapter 1 presents a determinant-free method of solving linear equations (in terms of "echelon systems") over the

field of real numbers, and concludes by defining "field" and pointing out that the method works in every field. The general definitions in Chapter 2 (e.g., dimension) are applied to matrices in Chapter 3 (e.g., to row rank and column rank). Most of this is, perforce, done rather computationally; the associative law for matrix multiplication, for instance, is proved by explicit examination of the appropriate double sums. Chapter 4 introduces the determinant as a function of n vectors. Chapter 5 has the standard material (including Sylvester's law of inertia) on quadratic and Hermitian forms. The abstract point of view appears in Chapter 6; groups and rings are defined and applied to linear transformations. Chapter 7 treats similarity, and Chapter 8 applies the results to the solution of the problem of unitary equivalence for normal matrices. The real case is handled directly (without complexification); the principal axis theorem and some extremal properties of characteristic values are derived.

P. R. Halmos (Chicago, Ill.).

Stojaković, Mirko. Sur les déterminants des matrices rectangulaires. Bull. Soc. Roy. Sci. Liège 21, 303–305 (1952).

Stojaković, Mirko. Determinanten rechteckiger Matrizen. Bull. Soc. Math. Phys. Serbie 4, nos. 1–2, 9–23 (1952). (Serbo-Croatian. German summary)

Let $\mathfrak{a}$ be a matrix with r rows and s columns. For $n \leqq \min(r, s)$ the author defines $\det_n \mathfrak{a}$ as the sum of all n-by-n determinants formed out of $\mathfrak{a}$ without altering the sequence of rows or columns, and studies its properties. He defines $\det \mathfrak{a}$ as $\det_n \mathfrak{a}$ for $n = \min(r, s)$, and derives many determinantal properties, including a Laplace expansion with an ingenious sign convention. The linear dependence of the rows or columns of $\mathfrak{a}$ is related not to $\det \mathfrak{a}$, but to the norm $N(\mathfrak{a}) = \{\det(\mathfrak{a}\mathfrak{a}')\}^{\frac{1}{2}}$ $(r \leqq s)$ or $\{\det(\mathfrak{a}'\mathfrak{a})\}^{\frac{1}{2}}$ $(r \geqq s)$.

When $\det \mathfrak{a} \neq 0$, the author defines $\mathfrak{a}^{-1}$ in terms of cofactors. If $r < s$, $\mathfrak{a}\mathfrak{a}^{-1} = I$, while $\mathfrak{a}^{-1}\mathfrak{a}$ is the "extraordinary unit matrix" of Bjerhammar [Bull. Géodésique **1951**, 188–220; these Rev. **13**, 312]. Reviewer's note: when $\det \mathfrak{a} \neq 0$, $\mathfrak{a}^{-1}$ is one of Bjerhammar's inverses, which depend on parameters outside $\mathfrak{a}$; when $\det \mathfrak{a} = 0$, Bjerhammar's inverses may exist, while $\mathfrak{a}^{-1}$ does not.

G. E. Forsythe.

Rutledge, W. A. Quaternions and Hadamard matrices. Proc. Amer. Math. Soc. **3**, 625–630 (1952).

If H is an n-rowed such matrix whose elements are selected from the set of sixteen quaternions $\pm 1, \pm i, \pm j, \pm k$ and $H\bar{H}' = nI$, then H is called a quaternionic Hadamard matrix. There is then an equivalent diagonal normal form $\{d_1, \cdots, d_n\}$ of A. The author shows that if n is a product of distinct odd primes, we can take $d_i = 1$ for $i < t = \frac{1}{2}(n+1)$, d_t an odd primitive quaternion of norm n, $d_j = n$ for $t < j = n$.

A. A. Albert (Chicago, Ill.).

Duncan, D. G. On D. E. Littlewood's algebra of S-functions. Canadian J. Math. **4**, 504–512 (1952).

This paper concerns the plethysm of S-functions, i.e., the expression of $\{\lambda\} \otimes \{\mu\}$ as a sum of S-functions. In particular, it is devoted to a method of computing $\{m\} \otimes \{4\}$ from the formula

$$\{m\} \otimes \{4\} = \tfrac{1}{12}[\tfrac{3}{2}(t_1^2 + t_2)^2 - t_1^4 + 4t_1t_3 + 3t_4]$$

where $t_r = \{m\} \otimes S_r$. Various techniques are developed for simplifying the computation, but it depends fundamentally on the theorem that $\{m\} \otimes S_n = \sum \pm \{\lambda\}$, where $\{\lambda\}$ is summed for all partitions of mn which have a null n-core, the sign being $+$ or $-$ according as the sum of the leg lengths of the removed n-hooks is even or odd.

This theorem is a particular case of a more general theorem proved by the reviewer [Proc. Roy. Soc. London. Ser. A. **209**, 333–353 (1951), see p. 351; these Rev. **14**, 243]. In fact the methods of this paper are at once less general and more complicated than a procedure suggested by the reviewer [loc. cit., p. 352]. It is only fair to state, however, that although the reviewer's paper antedates this, it was not in print when this paper was written. *D. E. Littlewood.*

Forbat, N. Démonstration élémentaire d'un théorème de Sylvester sur les formes quadratiques. Mathesis **61**, 256–258 (1952).

Abstract Algebra

Löwig, Henry F. J. On the properties of freely generated algebras. J. Reine Angew. Math. **190**, 65–74 (1952).

A freely generated algebra is understood to be without laws: two functions of elements in the algebra are equal only if they are formally identical. This paper formulates rigorously Birkhoff's definition of a freely generated algebra in the case where the number of operations, and their ranks, are unrestricted. Certain properties are derived.

In a highly formalized notation, the concepts of subalgebra, generating set, homomorphism are developed: e.g., Theorem 1.10: the elements on which two homomorphisms agree form a subalgebra. Next (§2) the essential definitions. A set Q in the algebra A, with closure $\bar{Q}$, is free if no function over $\bar{Q}$ lies in Q, and no element has distinct representations as a function of elements from $\bar{Q}$. Then a subset of a free set is free; and freely generated $\bar{Q}$ has a unique free basis Q. It is established, for freely generated A, that every mapping of the basis defines a homomorphism, and an isomorphism if the image of the basis is free. Other results deal with the (formal) dependence of elements on others, and with cardinality relations. *R. C. Lyndon* (Princeton, N. J.).

Rédei, L. Über gewisse Ringkonstruktionen durch schiefes Produkt. Acta Math. Acad. Sci. Hungar. **2**, 185–189 (1951). (Russian summary)

Let P be a ring with elements $\alpha, \beta, \cdots$ and R a ring with elements $a, b, \cdots$. (It is sufficient to assume R as a system with two compositions $a+b$ and ab.) The author raises the question: Under what conditions does the set of pairs (a, α) with the compositions $(a, \alpha)+(b, \beta) = (a+b, \alpha+\beta)$ and either

(i) $$(a, \alpha)(b, \beta) = (ab, b\alpha + a\beta + \alpha\beta)$$

or

(ii) $$(a, \alpha)(b, \beta) = (ab, \alpha b + a\beta + \alpha\beta)$$

form a ring? By an elementary discussion he proves that the obviously necessary conditions which express that P is an R-module (left or two-sided, respectively) are also sufficient. If R is the ring of integers, one obtains the well-known method of adjoining a one-element to a ring.

K. A. Hirsch (London).

Zemmer, Joseph L., Jr. On the subalgebras of finite division algebras. Canadian J. Math. **4**, 491–503 (1952).

Let A be a commutative not necessarily associative division algebra, with unit element, of order n over a Galois field $F = GF(q^k)$ $(q > 2)$, and let A_m denote a subalgebra of order m over F. It is not known whether m always divides n,

as in the theory of associative division algebras, but it is shown that m cannot be even if n is odd. For n even, A_2 exists, is unique, consists of the roots in A of quadratic equations over F, and is isomorphic to $GF(q^{2k})$. In the case $n=4$, A has a basis $1, f, f^2, f^3$ with 1 and f^2 spanning A_2; this improves a theorem of Dickson [Trans. Amer. Math. Soc. **7**, 370–390 (1906), p. 381].

Let A be not necessarily commutative but otherwise as above, and let it be postulated that $G(A)$, the automorphism group of A relative to F, contains the cyclic group C_n. If an automorphism generating this C_n has its minimum function of degree n, then for every divisor m of n there exists a subalgebra A_m, such that [the reviewer questions whether this is proved] $G(A_m)$ contains C_m. If n divides q^k-1, then A has relative to F a cyclic basis; that is, a basis consisting of right powers $1(=e^0), e, e^2, \cdots, e^{n-1}$ where $e^ie^j=\phi_{ij}e^k$, $k\equiv i+j$ modulo n, $\phi_{ij}\varepsilon F$; which evidently again guarantees existence of A_m for every divisor m of n.

Existence of non-associative finite division algebras A^* satisfying the above postulate is shown by constructing a commutative A^* of order $2n$ over F, for which $G(A^*)$ is C_{2n}, where $2n$ is any even divisor of q^k-1; the method of construction derives from Dickson [ibid. **7**, 514–522 (1906)]. This A^* has a unique associative subalgebra of order n over F [the reviewer questions whether "unique" is proved].

I. M. H. Etherington (Edinburgh).

Wever, Franz. Über reduzierte freie Liesche Ringe. Math. Z. **56**, 312–325 (1952).

Let A be the free (noncommutative) ring of all polynomials in r generators $X_1, \cdots, X_r$ with rational coefficients. Then A is also a Lie ring relative to the operation $a\cdot b=ab-ba$ and contains a Lie subring of all polynomials in $X_1, \cdots, X_r$ defined relative to the product operation $a\cdot b$. We define A_m to be the submodule of A of all homogeneous polynomials of degree m and L_m the corresponding submodule of L. The author then defines an operator set O_m in the group ring of all permutations on n letters and shows that $A_m=\Omega_m(A_m)$. Let M be an ideal of L, M_m be the intersection of M and L_m. Then M is said to be completely invariant in L if every linear transformation on $X_1, \cdots, X_r$ carries every M_m into M_m, and every homomorphic mapping of L into L maps every M_m into the ideal generated by M_m. Then the difference ring $L-M$ is called a reduced free Lie ring. A polynomial $p(y_1, \cdots, y_n)$ with rational coefficients is called a rule if $p(a_1, \cdots, a_n)=0$ for every $a_1, \cdots, a_n$ of L. Then a Lie ring L is reduced free if every relation between the generators of L is a rule. The author studies the generation of all rules in a reduced free ring. The definitions originate in the theory of free groups. *A. A. Albert.*

Schenkman, Eugene. Infinite Lie algebras. Duke Math. J. **19**, 529–535 (1952).

The author studies infinite-dimensional Lie algebras L subject to the restriction: L has a nil radical N (i.e., union of all nilpotent ideals) such that $N^k=0$ and L/N is finite (dimensional). Most results require the additional restriction (F): L is locally finite, i.e., a finite set of elements generate a finite subalgebra, or (UF): L is uniformly locally finite, i.e., any n elements generate a subalgebra of dimension $\leqq\phi(n)$, where $\phi(n)$ depends only on n. Engel's theorem on Lie algebras with nilpotent adjoint representation extends. Levi's decomposition extends under hypothesis (F). Lie's theorem that the radical of a solvable algebra is nilpotent extends under hypothesis (UF). Among other results the author proves the following. Theorem: Assume that the center of L is zero and L satisfies (UF). Let $D_0, D_1, D_2, \cdots$ be the tower of derivation algebras of L. If Z is the center of $L^\omega=\bigcap_1^\infty L^k$, then, for $k=1, 2, \cdots$, D_k/Z is isomorphic to a subalgebra of $D(L^\omega)$, the derivation algebra of L^ω. Errata: p. 534, l. 11b: "L^ω" for "L"; l. 2b: "L^ω" for "L".

G. D. Mostow (Baltimore, Md.).

Hillman, Abraham. On the differential algebra of a single differential polynomial. Ann. of Math. (2) **56**, 157–168 (1952).

The principal result of this paper is a generalization of the following theorem of Ritt [Amer. J. Math. **60**, 1–43 (1938)]. Let A and B be partial differential polynomials over an abstract differential field in n (differential) indeterminates $y_1, \cdots, y_n$. Let B hold A. Let $\bar{A}$ be the sum of terms of A of lowest (or highest) degree in the y_i and their derivatives and let $\bar{B}$ be the corresponding sum for B. Then $\bar{B}$ holds $\bar{A}$.

The generalization consists in introducing a scheme for ranking terms which is more general than that based on degree, and then showing that terms of least rank are related as in Ritt's theorem. This generalization is used to obtain generalizations of known results on the representation of one partial differential polynomial as a polynomial in another. There is also given a sufficient condition for a singular solution of a partial differential polynomial to be contained in its general solution. *H. Levi* (New York, N. Y.).

Theory of Groups

Popova, Hélène. Sur les vecteurs dérivés des quasigroupes unis. C. R. Acad. Sci. Paris **235**, 1360–1362 (1952).

A quasigroup of finite order n is called plain (uni) if it is simple and has no subquasigroups other than itself. In effect, a derived vector of a plain quasigroup Q means a subset every element of which has the same logarithmetic [cf. Popova, same C. R. **234**, 1936–1937 (1952); these Rev. **13**, 906]. The range (portée) r of Q means the greatest size of such a subset. It is shown that r divides n and that the elements of Q fall into n/r derived vectors with r elements in each. It follows that the order of the logarithmetic of Q does not exceed $n^{n/r}$. *I. M. H. Etherington.*

Schwarz, Štefan. On semigroups having a kernel. Czechoslovak Math. J. **1(76)** (1951), 229–264 (1952) = Čehoslovack. Mat. Ž. **1(76)** (1951), 259–301 (1952).

Let S be a semigroup having a non-vacuous Suschkewitsch kernel $\mathfrak{n}$ (intersection of all the two-sided ideals of S). A left ideal L of S is called simple if $L\supset\mathfrak{n}$ (proper inclusion) and if there exists no left ideal between L and $\mathfrak{n}$; simple right and two-sided ideals are defined similarly. This reduces to the usual notion of minimal ideal when S contains a zero element 0, in which case $\mathfrak{n}=(0)$. The author extends to the general case earlier results of A. Suschkewitsch [Math. Ann. **99**, 30–50 (1928)], D. Rees [Proc. Cambridge Philos. Soc. **36**, 387–400 (1940); these Rev. **2**, 127], himself [Sborník Prác Prírodovedeckej Fakulty Slovenskej Univerzity v Bratislave no. 6 (1943); these Rev. **10**, 12], and the reviewer [Amer. J. Math. **71**, 834–844 (1949); these Rev. **11**, 327].

The results of the concluding section (10) of the paper are new even for the case $\mathfrak{n}=(0)$. As in the earlier paper cited

above, an ideal is called $\mathfrak{n}$-potent if some power of it is contained in $\mathfrak{n}$, and the radical of S is the sum of all the $\mathfrak{n}$-potent ideals of S. Let $\mathfrak{L}^{(0)}(\mathfrak{L}^{(1)})$ be the sum of all the simple $\mathfrak{n}$-potent (non-$\mathfrak{n}$-potent) left ideals of S. Then $\mathfrak{L}=\mathfrak{L}^{(0)}\cup\mathfrak{L}^{(1)}$ is a two-sided ideal of S, and $\mathfrak{L}^{(0)}$ is the radical of the semigroup $\mathfrak{L}$. Every simple non-$\mathfrak{n}$-potent left ideal of S is contained in a simple non-$\mathfrak{n}$-potent two-sided ideal of S if and only if $\mathfrak{L}^{(1)}$ is a two-sided ideal of S, in which case $\mathfrak{L}^{(1)}$ is the sum of simple non-$\mathfrak{n}$-potent two-sided ideals. $\mathfrak{R}^{(0)}$, $\mathfrak{R}^{(1)}$, $\mathfrak{M}^{(0)}$, $\mathfrak{M}^{(1)}$ are defined similarly for the simple right and two-sided ideals of S, and conditions are given under which relations like $\mathfrak{L}^{(1)}=\mathfrak{R}^{(1)}$, $\mathfrak{L}^{(1)}=\mathfrak{M}^{(1)}$, etc. hold. The author assumes throughout Section 10 that the radical of S is itself $\mathfrak{n}$-potent, but the reviewer feels that this assumption is not necessary. [Correction to Theorem 8.3: one must assume that $L^{(M)}$ and $L^{(S)}$ are non-$\mathfrak{n}$-potent, and this is just what is needed in the proof of Theorem 10.2, upon which all of Section 10 depends. "Remark 3" applies only if M is assumed to be simple.] *A. H. Clifford* (Baltimore, Md.).

Teissier, Marianne. Sur quelques propriétés des idéaux dans les demi-groupes. C. R. Acad. Sci. Paris **235**, 767–769 (1952).

This note continues an earlier one by the same author [same C. R. 234, 386–388 (1952); these Rev. **13**, 620]. Let D be a demi-group having a Suschkewitsch kernel $\mathfrak{s}$ which is the sum of minimal left ideals of D. As in the earlier paper, a left ideal $\mathfrak{G}'$ of D is called a minor ideal if every proper left subideal of $\mathfrak{G}'$, but not $\mathfrak{G}'$ itself, is contained in $\mathfrak{s}$. Let $\mathfrak{G}'$ be a minor left ideal of D such that $\mathfrak{G}'^2=\mathfrak{G}'$ and such that $\mathfrak{G}'\cap\mathfrak{s}$ is the sum of a finite number of minimal left ideals $\mathfrak{G}_i$. Then the set $\mathfrak{R}$ of all elements x of $\mathfrak{G}'$ such that $\mathfrak{G}'x=\mathfrak{G}'$ is a non-vacuous subdemi-group of D. Multiplication by x induces a permutation $\mathfrak{G}_i\rightarrow\mathfrak{G}_ix$ of the $\mathfrak{G}_i$, and $\mathfrak{R}$ is thereby homomorphic to a group of permutations. If the part $\mathfrak{C}$ of $\mathfrak{G}'$ not in $\mathfrak{s}$ is a group, then $\mathfrak{R}=\mathfrak{C}$. The theory of representations of a group by permutations is applied to $\mathfrak{C}$. This is generalized to the case where $\mathfrak{G}'D$ contains a finite number of minimal right ideals. *A. H. Clifford.*

Greco, Donato. I gruppi finiti che sono somma di quattro sottogruppi. Rend. Accad. Sci. Fis. Mat. Napoli (4) **18** (1951), 74–85 (1952).

The author proves that a finite group G can be written as the sum of precisely four proper subgroups if and only if it possesses a normal subgroup N with a factor-group G/N of one of the following four types: (i) non-Abelian of order 6, (ii) of order 8, but not cyclic, nor quaternion, (iii) of order 9, not cyclic, (iv) of order 18 and the type which arises when a non-cyclic group of order 9 is extended by an automorphism of order 2 which carries each of the two generators into its inverse. The proof is elementary, but requires a large number of tedious case distinctions. The corresponding problem for three subgroups was solved by G. Scorza [Boll. Un. Mat. Ital. (1) **5**, 216–218 (1926)]. *K. A. Hirsch.*

Trofimov, P. I. On the influence of the number of all classes of noninvariant conjugate subgroups upon the properties of a finite group. Doklady Akad. Nauk SSSR (N.S.) **86**, 1075–1076 (1952). (Russian)

Let G be a finite group of order g; let $\tau(G)$ be the number of distinct prime divisors in g; let $\rho(G)$ be the number of classes of non-invariant conjugate subgroups in G. For $\rho(G)=1, 2$, we have $\tau(G)\leqq\rho(G)+1$. For $\rho(G)=3, 4$, we have $\tau(G)\leqq\rho(G)$. These bounds can be reduced by 1 for "special" groups. For $\rho(G)\geqq5$, we have $\tau(G)\leqq\rho(G)-2$. A more special result on structure is the following. Suppose $\tau(G)=\rho(G)=3$, g not squarefree; or suppose $\tau(G)=\rho(G)=4$. Then $g=\prod p_i^{\alpha_i}$, where for $i>1$ we have $\alpha_i<3$; the Sylow subgroups corresponding to p_i, as well as their subgroups, are invariant in G; the other Sylow subgroups are cyclic; they are not invariant in G, but all their subgroups are. No proofs are given. *J. L. Brenner* (Pullman, Wash.).

Honda, Kinya. On finite groups, whose Sylow groups are all cyclic. Proc. Japan Acad. **25**, 154–159 (1949).

A hypercyclic group is a finite group all the Sylow subgroups of which are cyclic. It is known [Zassenhaus, Lehrbuch der Gruppentheorie, Teubner, Leipzig and Berlin, 1937, pp. 138–139] that such groups are solvable with cyclic commutator and factor-commutator groups and that the structure of such groups can be given by certain defining relations. Vincent [Comment. Math. Helv. **20**, 117–171 (1947); these Rev. **9**, 131] has showed that the finite rotation groups without fixed points of the sphere S^{4k+1} are hypercyclic, so that non-trivial examples of such groups are known. The author announces results which are given in part here. Proofs are omitted and are to appear elsewhere, according to the author. The paper is plagued with numerous misprints, inverted type and omissions, not only of letters and symbols, but also of parts of the statements of the results. It is known that hypercyclic groups can be decomposed into a sort of generalized direct product of its Sylow subgroups. Each initial set of factors contains a portion of the centralizer of the next factor. These portions are called the foundation groups of the hypercyclic group. Each has an index in the set of initial factors in which it is included. Such sets of indices determine classes (L-classes) of hypercyclic groups with closely related lattices of subgroups. Using the Euler totient function, the author is able to enumerate the number of distinct (to within isomorphism) groups in an L-class. In a hypercyclic group, subgroups of the same order are conjugate. *F. Haimo.*

Gaschütz, Wolfgang. Zur Erweiterungstheorie der endlichen Gruppen. J. Reine Angew. Math. **190**, 93–107 (1952).

Definitions. Throughout, G denotes a finite group which is an extension of an Abelian group A by means of $H\cong G/A$. The transformations of A by elements of G induce a homomorphism Γ of H onto a subgroup $\Gamma(H)$ of the group of automorphisms of A. If G possesses a subgroup $\bar{G}$ such that $G=A\bar{G}$ and $A\cap\bar{G}\subset A$ $[A\cap\bar{G}=1]$, then G is said to split ("zerspaltet") [split completely ("zerfällt")] over G. (In the terminology of P. Hall [same J. **182**, 206–214 (1940); these Rev. **2**, 125] G is a partially complemented [complemented] extension of A.) Let α be a group of automorphisms of A. A splitting extension $G=A\bar{G}$ is called α-invariant if beyond $A\cap\bar{G}\subset A$ even $N_\alpha(A\cap\bar{G})\subset A$, where $N_\alpha(K)$ is the join of the subgroups conjugate to K under the automorphisms of α. A group G is called p-perfect if it possesses no normal subgroup with a p-group as factor-group.

The author proves two reduction theorems which simplify the search for (completely) splitting extensions G of A and illustrates their scope by applications to a number of special cases. Let A_{p_i}, $i=1, 2, \cdots, r$, be the Sylow subgroups of A and A_{p_i}' their direct product with the exception of A_{p_i}, so that $A=A_{p_i}\times A_{p_i}'$. First Reduction Theorem: G splits [splits completely] over A if and only if, for some [every] index i, G/A_{p_i}' splits over A/A_{p_i}'. By this theorem the search for splitting extensions over Abelian groups is reduced to that over Abelian p-groups. The second theorem

refers to the factor-group $H=G/A$. Second Reduction Theorem: G splits over the Abelian p-group A if and only if a p-Sylow subgroup of G is a $\Gamma(H)$-invariant splitting over A. The group G splits completely over A if and only if a p-Sylow subgroup splits completely over A.

From these theorems the author deduces the following results. Theorem 3: G splits completely over A if H possesses a nilpotent normal subgroup N such that no element $\neq 1$ of A remains invariant under $\Gamma(N)$ (generalisation of results of H. Bergström [Math. Nachr. 1, 350–356 (1948); these Rev. 11, 157]). Theorem 4: G splits completely over the Abelian p-group A if H is p-perfect, if the p-Sylow subgroups of H are cyclic, and if H is mapped by Γ onto the identical automorphism of A. Theorem 5. If $p\neq 2$, then G splits completely over the cyclic p-group A if a p-Sylow subgroup of H is mapped isomorphically by Γ. Theorem 6. If A is a unique minimal Abelian normal subgroup of G, and if the factor-group G/A contains an Abelian normal subgroup whose order is prime to the order of A, then G splits completely over A. (This theorem is due to Ore [Duke Math. J. 5, 431–460 (1939)].) Theorem 7. An extension G over a p-perfect group N (not Abelian!) by means of a p-group splits completely if the p-Sylow subgroups of N are Abelian. Theorem 8. A group G whose Sylow subgroups are all elementary-Abelian splits completely over every normal subgroup N. *K. A. Hirsch* (London).

Fuchs, L. The Zappa extension of partially ordered groups. Nederl. Akad. Wetensch. Proc. Ser. A. 55 = Indagationes Math. 14, 363–368 (1952).

In a recent paper [Acta Math. Acad. Sci. Hungar. 1, 118–124 (1950); these Rev. 13, 436] the author has shown that Schreier's extension theory can be carried over to partially ordered groups: given two partially ordered groups C and Γ, one can characterize all partially ordered groups G which contain a convex normal subgroup order-isomorphic to C with a factor-group order-isomorphic to Γ. The present note shows that the same procedure can be adopted in the case of the extension problem for the so-called Zappa-Szép product (for the definition and simplest properties see a series of papers by several authors reviewed in these Rev. 14, 13). *K. A. Hirsch* (London).

Zappa, Guido. Sui gruppi p-supersolubili. Rend. Accad. Sci. Fis. Mat. Napoli (4) 17 (1950), 328–339 (1951).

Finite groups are called supersoluble if all their chief factors are of prime order (a). It is well known [Wendt, Math. Ann. 55, 479–492 (1902)] that the derived group of a supersoluble group is nilpotent. In addition, Ore has proved [Duke Math. J. 5, 431–460 (1939)] that the following properties are characteristic for supersoluble groups: (b) the group and all its subgroups possess subgroups of all possible orders; (c) all complete chains in the group have prime indices; (d) any two complete chains are conformal. The concept of a soluble group has been extended by Čunihin to that of a p-soluble group, i.e., one in which the composition factors are all of order p or prime to p [see e.g. Mat. Sbornik N.S. 25(67), 321–346 (1949) = Amer. Math. Soc. Translation no. 72 (1952); these Rev. 11, 495; 14, 131]. In the present paper the author investigates what happens to the equivalence of properties (a)–(d) if supersoluble groups are replaced by p-supersoluble groups. The corresponding properties are then: $(\bar{a})$ all chief factors are of order p or prime to p; $(\bar{b})$ the group and all its subgroups have subgroups of all possible p-power indices; $(\bar{c})$ all complete chains in the group have indices p or prime to p; $(\bar{d})$ any two complete chains are p-conformal, i.e., the chains can be put into $1-1$ correspondence such that the indices prime to p are equal. The results of the paper are as follows: $(\bar{a})$ implies $(\bar{b})$, $(\bar{c})$, and $(\bar{d})$. But $(\bar{b})$ does not imply $(\bar{a})$, even if the group is assumed to be p-soluble. If p is the smallest prime factor of the order of the group, then $(\bar{a})$ and $(\bar{b})$ are again equivalent. $(\bar{c})$ implies $(\bar{a})$ if the group is assumed to be p-soluble. The author conjectures that groups with property $(\bar{c})$ are always p-soluble. Lastly $(\bar{d})$ implies $(\bar{c})$ if the group is p-soluble. Some of the author's results overlap slightly with those of Čunihin (loc. cit.), but the Russian papers were not accessible to the author. *K. A. Hirsch.*

Zappa, Guido. Determinazione degli elementi neutri nel reticolo dei sottogruppi di un gruppo finito. Rend. Accad. Sci. Fis. Mat. Napoli (4) 18 (1951), 22–28 (1952).

The author solves problem 35 of G. Birkhoff's Lattice Theory [Amer. Math. Soc. Colloq. Publ., v. 25, rev. ed., New York, 1948, p. 98; these Rev. 10, 673]: to characterize the neutral elements of the subgroup lattice $L(G)$ of a finite group G. The answer is as follows. Let G be of order g, and let A be a subgroup of G, of order n. Let l be the product of those prime factors of g which divide n to the same power as g, h the product of those which divide n, but to a lower power than g, m those which do not divide n, and k the l.c.m. of n and h so that $g=mhl$, $n=kl$. Then A is a neutral element of the subgroup lattice $L(G)$ if and only if $G=MH\times L$ and $A=K\times L$, where M, H, L, K are of orders m, k, l, k respectively, and where M is elementwise permutable with A, H is cyclic or the direct product of a cyclic group of odd order and a generalized quaternion group, K is cyclic and, in case H is not cyclic, k is not divisible by 4. The main tools of the author are his theory of homotropisms and hemitropisms [Giorn. Mat. Battaglini (4) 2(78), 182–192 (1949); 4(80), 80–101 (1951); these Rev. 11, 322; 12, 800] and G. Birkhoff's characterization of neutral elements in an arbitrary lattice [loc. cit., p. 28]. *K. A. Hirsch.*

Gorčinskiĭ, Yu. N. Groups with a finite number of classes of conjugate elements. Mat. Sbornik N.S. 31(73), 167–182 (1952). (Russian)

G. Higman, B. H. Neumann and H. Neumann [J. London Math. Soc. 24, 247–254 (1949); these Rev. 11, 322] have shown that any group without elements of finite order $\neq 1$ can be embedded in a group in which all elements $\neq 1$ are conjugate. The present author proves by similar methods (free product with amalgamated subgroups) that if a group G has only elements of s distinct prime orders (apart from 1 and, possibly, elements of infinite order), then, for any natural number $n>s+1$, G can be embedded in a group $\bar{G}$ with precisely n classes of conjugate elements. He shows further that if G and H are two non-isomorphic, non-cyclic groups in which all elements $\neq 1$ have order p (p a prime), then the groups $\bar{G}$ and $\bar{H}$ cannot be isomorphic. It follows that the set of all non-isomorphic countable groups with precisely n classes of conjugates has, for any natural number $n\geqq 3$, the power of the continuum. *K. A. Hirsch.*

Gorčinskiĭ, Yu. N. Periodic groups with a finite number of classes of conjugate elements. Mat. Sbornik N.S. 31(73), 209–216 (1952). (Russian)

This is a sequel to the author's recent paper on groups with a finite number of conjugate classes [see the preceding review] and contains some elementary observations on the case of periodic groups. If G is an arbitrary periodic group with a fixed number n of classes of conjugates, then the

orders of the elements of G are bounded by a number $k(n)$, depending on n only. It follows that there exist periodic groups with exactly s layers which cannot be embedded in periodic groups with s classes of conjugates. This provides a negative answer to a question raised by the author (loc. cit.). If p is the smallest prime divisor of the order of G and if G contains cyclic groups of order p^a, then $a(p-1)\leqq n-1$. This theorem enables the author to enumerate the (few) types of periodic groups with exactly three classes of conjugates. *K. A. Hirsch* (London).

Suprunenko, D. A. Irreducible nilpotent matrix groups of prime degree. Mat. Sbornik N.S. 31(73), 353–358 (1952). (Russian)

Let Ω be an algebraically closed field, p a prime number, $GL(p, \Omega)$ the full linear group of degree p over Ω, M the multiplicative group of Ω. The author proves the following three theorems. 1. If p is not the characteristic of Ω, then there exist (finitely) nilpotent irreducible matrix groups over Ω whose centre is equal to M, with an arbitrary preassigned length of the upper central series. 2. Let Γ and Γ' be nilpotent irreducible matrix groups of degree p over Ω with centre M, and let l, l' be the lengths of the upper central series of Γ, Γ' respectively. If $l=l'$, then Γ and Γ' are conjugate in $GL(p, \Omega)$; if $l>l'$, then Γ contains a subgroup conjugate to Γ' in $GL(p, \Omega)$. 3. The index of the centre of a nilpotent irreducible matrix group of degree p over Ω is equal to p^l, where l is the length of the upper central series. (The restriction on the centre in theorems 1 and 2 is not very serious, since for any nilpotent irreducible group Γ over Ω the group ΓM is also nilpotent and has centre M.) *K. A. Hirsch* (London).

Ado, I. D. On the theory of characters of finite groups. Doklady Akad. Nauk SSSR (N.S.) 50, 11–14 (1945). (Russian)

Let K be a field (presumably of characteristic 0), ϵ a primitive nth root of unity. Let A, B be the automorphisms given by $\epsilon\rightarrow\epsilon^i$ and ϵ^j. We consider a finite group G and an irreducible representation Γ of it in $K(\epsilon)$. The automorphisms give rise to new representations Γ^A and Γ^B. The author proves that the number of Γ's such that Γ^A is equivalent to Γ^B coincides with the number of conjugate classes of elements a in G satisfying $a^i=a^j$. Two further theorems give related formulas concerning characters. *I. Kaplansky* (Chicago, Ill.).

Ado, I. D. On subgroups of the countable symmetric group. Doklady Akad. Nauk SSSR (N.S.) 50, 15–17 (1945). (Russian)

Let G be the group of all finite permutations of the integers (i.e., all permutations moving only a finite number of elements). Let H be an infinite subgroup of G. The author proves that the normalizer of any element of H is infinite. Two immediate corollaries are drawn. If a subgroup K of G has all its proper subgroups finite, then K itself is finite; if K has the descending chain condition on subgroups it is finite. *I. Kaplansky* (Chicago, Ill.).

Iwahori, Nagayoshi. Non-representability of real general linear groups in higher dimensional Lorentz groups. Sci. Papers Coll. Gen. Ed. Univ. Tokyo 2, 13–23 (1952).

Let $GL(n, F)$ be the general linear group of degree n over the field F (assumed to be of characteristic 0 or $p\neq 0$) and let $O(m, k, F)$ be the group of linear transformations of degree $n+k$ leaving the form $x_1^2+\cdots+x_m^2-x^2_{m+1}-\cdots-x^2_{m+k}$ invariant. The author asks: which groups $GL(n, F)$ are isomorphic to subgroups of $O(m, k, F)$? (It is easily shown, for example, that $GL(n, F)$ can be mapped isomorphically into $O(n, n, F)$.) The author proves that if $n\geqq 3$, there exists no continuous one-one homomorphism of $GL(n, R)$ into $GL(m, 1, R)$ where R is the field of reals and m any integer >0. *P. A. Smith* (New York, N. Y.).

Murnaghan, F. D. On the invariant theory of the classical groups. Proc. Nat. Acad. Sci. U. S. A. 38, 966–973 (1952).

The principal results are inductive methods of evaluating plethysms of S-functions in the form $\{m\}\otimes\{j\}$, $\{1^m\}\otimes\{j\}$, $\{m\}\otimes\{1^j\}$, $\{1^m\}\otimes\{1^j\}$.

To obtain $\{m\}\otimes\{j\}$ put

$$\{m-1\}\otimes\{1^j\}=A_j,$$

$$\{m-1\}\otimes\{1^{j-1}\}\{m\}-A_j\{1\}=A_{j-1},$$

$$\{m-1\}\otimes\{1^{j-2}\}(\{m\}\otimes\{2\})-A_{j-1}\{1\}-A_j\{1^2\}=A_{j-2},$$

etc., only those terms being considered in A_{j-r} which correspond to partitions into no more than $j-r$ parts. Then the S-functions appearing in $\{m\}\otimes\{j\}$ which correspond to partitions into exactly j nonzero parts are obtained from the partitions in A_j by adding 1 to each of the j parts. Similarly for $j-1$ parts add one to each of $j-1$ parts in the partitions of A_{j-1}, and so on. A similar result holds for $\{m\}\otimes\{1^j\}$. The expansions for $\{1^m\}\otimes\{j\}$, $\{1^m\}\otimes\{1^j\}$ are obtained by the theorem of conjugates. No proof is given. An equivalent result has been given, and proved, by M. J. Newell [Quart. J. Math., Oxford Ser. (2) 2, 161–166 (1951); these Rev. 13, 312]. By using Newell's modification rules for the orthogonal and symplectic groups the author obtains corresponding expansions for these groups. *D. E. Littlewood* (Bangor).

Heinz, C. Unbedingte und bedingte Invarianten bei Gruppen und bei von Gruppen umbeschriebenen Scharen von Transformationen. Math. Ann. 125, 32–48 (1952).

The present paper deals with groups and other families of transformations in n variables, and generalizes properties that proved to be useful in investigations of the group-theoretical aspects of some geometric objects [F. Kraus, Math. Ann. 96, 688–718 (1927); C. Heinz, Dissertation, Aachen, 1943]. One of the goals here is the construction of a complete set of invariants $J(x)$ connected with a family of transformations

$$y^i=f^i(x^j, \alpha^q);\quad i, j=1, \cdots, n;\quad q=1, \cdots, r,$$

and then to obtain the full group under which the $J(x)$ remain invariant. A distinction is made between "unbedingte" and "bedingte" invariants, the latter being the invariants under those transformations in which the x are restricted by conditions $B(x)=0$. Since the results are derived under the assumption that the x and the α lie in regions in which the rank of $\|\partial f/\partial\alpha\|$ is maximal, all properties are to be understood as being only locally valid. *A. Nijenhuis* (Princeton, N. J.).

Tits, J. Sur les groupes doublement transitifs continus. Comment. Math. Helv. 26, 203–224 (1952).

En continuant des recherches précédentes [Tits, Compositio Math. 9, 85–96 (1951); ces Rev. 12, 673; Freudenthal, Proc. Nederl. Akad. Wetensch. Ser. A. 54, 288–294 (1951); ces Rev. 13, 432], l'auteur démontre qu'un groupe doublement transitif dans un espace E localement compact,

non totalement discontinu et à premier axiome de dénombrabilité est essentiellement identique au groupe des transformations entières linéaires dans le corps réel, complexe, ou quaternionien. L'auteur fait voir que le groupe donné engendre un presque-corps qui d'après Kalscheuer [Abh. Math. Sem. Hansischen Univ. **13**, 413–435 (1940); ces Rev. **1**, 328] doit être un des trois corps classiques. En fait l'auteur envisage des systèmes plus généraux, appelés pseudo-corps, en remplaçant l'associativité de l'addition par le postulat plus faible de l'existence d'une fonction $\rho(b, c)$ telle que $(a+b)+c=\rho(b, c)a+(b+c)$. Un pseudo-corps topologique qui satisfait aux mêmes conditions topologiques que E est essentiellement un des trois corps classiques (quasi-généralisation d'un théorème de van Dantzig-Pontrjagin). Dans le cas d'un presque-corps on peut se passer de l'hypothèse de dénombrabilité. Enfin l'auteur envisage des groupes localement compact, connexe, de dimension finie satisfaisant au deuxième axiome de dénombrabilité et possédant un automorphisme involutif à point fixe isolé dans l'unité. Ces groupes sont nécessairement abéliens. *H. Freudenthal.*

Gel'fand, I. M., and Graev, M. I. Unitary representations of real simple Lie groups. Doklady Akad. Nauk SSSR (N.S.) **86**, 461–463 (1952). (Russian)

The authors initiate an investigation of the unitary irreducible representations of the real simple Lie groups, making strong use of the complex form of the group, for which the representations are known. This note is concerned chiefly with the real unimodular group in an arbitrary number n of dimensions; the case $n=2$ has been treated by Bargmann [Ann. of Math. (2) **48**, 568–640 (1947); these Rev. **9**, 133]. The main result (known for the case $n=2$ from Bargmann's work) is that the members of a certain explicitly given class of representations, designated as the principal series and related to the corresponding series for the complex unimodular group, are all irreducible. *I. E. Segal.*

Segal, I. E. Hypermaximality of certain operators on Lie groups. Proc. Amer. Math. Soc. **3**, 13–15 (1952).

Let G be a connected Lie group, let $\mathcal{E}$ be the free enveloping algebra of $\mathfrak{L}$, the Lie algebra of G, and let U be a strongly continuous unitary representation of G on the complex Hilbert space $\mathfrak{H}$. Let D consist of all sums of vectors of the form $\int U(a)xf(a)\,da$ with $x\in\mathfrak{H}$, where f is a function of class C^∞ on G vanishing outside a compact set. An element of $\mathcal{E}$ is called symmetric if it is invariant under the unique real linear operation on $\mathcal{E}$ which takes any monomial $\alpha X_1\cdots X_r$ (α complex, $X_i\in\mathfrak{L}$) into $(-1)^r\bar{\alpha}X_r\cdots X_1$. The author proves the theorem: If p is a symmetric polynomial in the center of $\mathcal{E}$, then $dU(p)$ is essentially hypermaximal symmetric on the domain D. As a consequence, if the weakly closed algebra of operators generated by the $U(a)$, $a\in G$, is a factor, then $dU(p)$ is the operator of multiplication by a scalar. *G. D. Mostow* (Baltimore, Md.).

Satake, Ichiro. On a theorem of E. Cartan. J. Math. Soc. Japan **2**, 284–305 (1951).

The author offers some improvements in an exposition of Cartan's result that the semi-group of highest weights of a complex semi-simple Lie algebra L is isomorphic to the direct product $I^+\times\cdots\times I^+$ (n times), where I^+ is the additive semi-group of non-negative integers and $n=\text{rank } L$. In an appendix he gives the following sharpened version of a result of Weyl: Let G be a compact connected semi-simple Lie group, H a Cartan subgroup, G^r the set of regular elements, $H^r=G^r\cap H$, and C a connected component in H^r. Then $C\times G/H$ is a simply connected covering space of G^r with respect to the mapping $(h, Hg)\to g^{-1}hg$. Every Deck-transformation can be identified with an element of the Cartan group of the root system. As a consequence the fundamental group of G^r (and hence of G) is finite. *G. D. Mostow* (Baltimore, Md.).

★Chevalley, Claude. Théorie des groupes de Lie. Tome II. Groupes algébriques. Actualités Sci. Ind. no. 1152. Hermann & Cie., Paris, 1951. vii+189 pp.

The great contribution of Sophus Lie to the theory of continuous groups was his very fruitful method of investigating a group via its Lie algebra. A formulation of the connections between a group and its Lie algebra involves the limit concept in a seemingly essential way—for example, the function e^x plays a cardinal role. On the other hand, an abstract Lie algebra can be defined over arbitrary ground fields. The juxtaposition of these two facts poses the interesting question: to what extent can the correspondence between the Lie group and Lie algebra structures be generalized? This problem was taken up jointly by the author and H. F. Tuan [Proc. Nat. Acad. Sci. U. S. A. **31**, 195–196 (1945); these Rev. **7**, 4], and in volume 2 (of his projected series of six volumes) of "Theory of Lie Groups", the author describes an extension of the classical mechanism to algebraic groups of endomorphisms (i.e., linear transformations) of a linear space over a field K of characteristic 0. As an application, the author can determine all subgroups of the multiplicative group of a given finite extension L of the field K which, considered as groups of transformations of the linear structure of L over K, are algebraic groups. Concerning this the author comments "Il n'est pas interdit d'espérer que ces groupes joueront un certain rôle dans la théorie arithmétique du corps L"—this being the case when K is an algebraic number field and L a cyclic extension of K. In case the characteristic of K is not 0 but K is infinite, the author's definition of Lie algebra is still meaningful. However the one-to-one correspondence between algebraic Lie groups and algebras is lost.

Apart from constituting a generalization of the classical theory of Lie groups, the theory yields new results on real and complex algebraic Lie groups. For example, there is a complete discussion of the question: Is the image of an algebraic Lie group under a rational representation (cf. below for definition) algebraic? Also, there is a very simple proof of the theorem that any algebraic Lie group is the group of all endomorphisms keeping certain rational functions invariant—from which one can deduce that a normal algebraic subgroup of an algebraic group is the kernel of a rational representation. Finally, there is a characterization of algebraic Lie algebras generalizing and extending results of Maurer on complex Lie algebras.

The outstanding feature of the exposition is the elegant organization of ideas. The basic definitions are chosen deftly, and each topic is developed with simple directness. Another feature is the meticulous treatment of details which are usually passed over lightly. The book is essentially self-contained and puts the theory on a clear-cut foundation. The first fifty pages or so are devoted to background material.

Once and for all, let K and L denote infinite fields, with $K\subset L$, U and V finite-dimensional linear spaces over K, $\mathcal{E}(U)$ the linear space of all endomorphisms of U, U^L the extension of U to a linear space over the field extension L of K, $\mathfrak{R}(U)$ the algebra of rational functions on U. Elements of $\mathfrak{R}(U)$ will be considered as elements of $\mathfrak{R}(U^L)$ in the natural way.

An algebraic group in $\mathcal{E}=\mathcal{E}(V)$ is a group consisting of all automorphisms in $\mathcal{E}$ on which some set of polynomial functions in $\mathcal{R}(\mathcal{E})$ vanish. The ideal of polynomial functions which vanish on a set $S\subset\mathcal{E}$ is called the ideal associated with S. If G is an algebraic group and $L\supset K$, then G^L is the smallest algebraic group in $\mathcal{E}^L$ which includes G. A generalized point of G is a point of G^L where L is any extension of K. If L and L' are field extensions of K and U is a linear space over K, an element $s'\in U^{L'}$ is called a specialization of $s\in U^L$ if and only if $P(s)=0$ implies $P(s')=0$ for any polynomial function P in $\mathcal{R}(U)$. A generic point of an algebraic group G is a generalized point such that every generalized point is a specialization of it. G has a generic point if and only if it is irreducible, i.e., its associated ideal is prime. The algebraic component of the identity of an algebraic group G is the irreducible subgroup of finite index (it exists and is unique).

A rational mapping of a linear space U into a linear space V is a mapping

$$a\rightarrow H_1(a)b_1+H_2(a)b_2+\cdots+H_n(a)b_n$$

where $H_1,\cdots,H_n$ are in $\mathcal{R}(U)$ and $b_1,\cdots,b_n$ are in V, and the domain is restricted as necessary. A rational mapping R of an irreducible algebraic group $G\subset\mathcal{E}(U)$ into $\mathcal{E}(V)$ is the restriction to G of a rational mapping $\bar{R}$ of $\mathcal{E}(U)$ into $\mathcal{E}(V)$ whose domain contains at least one point of G. A rational representation of an algebraic group G into $E(V)$ is a homomorphism whose restriction to G_1, the algebraic component of the identity, is a rational mapping defined at all points of G_1. If G is irreducible and R is a rational mapping of G into $\mathcal{E}(V)$ such that $R(xy)=R(x)\,R(y)$ whenever $R(x)$, $R(y)$, $R(xy)$ exist, then R is a rational representation.

The results on rational mappings and rational representations of algebraic groups and the induced mappings of the Lie algebras lie at the heart of the entire theory. For, from these follow the correspondences between the group and the Lie algebra structures just as in the classical theory. The role of the Lie algebra in clarifying the phenomena that can arise under rational mappings can best be appreciated by contrasting the results on rational representations of groups with the results on the induced representations of the Lie algebras. For example, the image of an algebraic Lie group under a rational representation need not be algebraic, except when the ground field is algebraically closed. However, the analogue for algebraic Lie algebras (over fields of characteristic 0) does not require the hypothesis of algebraic closure.

Let G be an algebraic group in $\mathcal{E}(V)$. The Lie algebra $\mathfrak{g}$ of G is defined essentially as the totality of right-invariant infinitesimal transformations of $\mathcal{E}(V)$ (i.e., infinitesimal left-translations) which are tangent to G except that these terms appear under algebraic aliases. Instead of "infinitesimal transformation", one sees "derivation" and the notion of tangency to an algebraic group is expressed by saying that a derivation keeps the associated ideal invariant. Elements of the Lie algebra are identified with elements of $\mathcal{E}(V)$.

Exponentials are made meaningful by use of formal power series. In view of the formula $e^x=\sum_{k=0}^{\infty}(k!)^{-1}x^k$, the discussion of exponentials can be carried out only by restricting K to fields of characteristic 0. Let $\mathfrak{T}$ denote the ring of formal power series in the letter T with coefficients in K, and let L denote the field of quotients of $\mathfrak{T}$. If X is in the Lie algebra of G, $\exp TX$ is a generalized point of G, namely a point of G^L. If $X_1,\cdots,X_d$ is a base for G, and L denotes the field of quotients of power series in letters $T_1,\cdots,T_d$ with coefficients from K, then $\exp(T_1X_1+\cdots+T_dX_d)$ is a point of G^L and is in fact a generic point of G.

The book closes with a characterization and description of algebraic Lie algebras, i.e., Lie algebras of algebraic groups. Every endomorphism X in $\mathcal{E}(V)$ can be expressed uniquely as $S+N$ where N is nilpotent and S is semi-simple, i.e., its minimal equation has no repeated factors. By $G(X)$ is meant the smallest algebraic group whose Lie algebra is X. By $\mathfrak{g}(X)$ is meant the Lie algebra of $G(X)$. Now $\mathfrak{g}(X)$ is $\mathfrak{g}(S)+\mathfrak{g}(N)$ (direct), $G(X)=G(S)\cdot G(N)$ (direct), and $\exp TX$ is a generic point of $G(X)$, T being an indeterminate. $\mathfrak{g}(N)$ consists of multiples of N alone. Let $x_1,\cdots,x_n$ be a base for V, and let $\operatorname{diag}(b_1,\cdots,b_n)$ denote the endomorphism Y of V such that $Yx_i=b_ix_i$ ($b_i\in K$, $i=1,\cdots,n$). Suppose $S=\operatorname{diag}(a_1,\cdots,a_n)$. Let Λ denote the totality of sets of integers $(e_1,\cdots,e_n)$ such that $a_1e_1+\cdots+a_ne_n=0$. Then $\mathfrak{g}(S)$ is the totality of endomorphisms $\operatorname{diag}(a_1',\cdots,a_n')$ such that $a_1'e_1+\cdots+a_n'e_n=0$ for all $(e_1,\cdots,e_n)$ in Λ. Also, $G(S)$ is the totality of automorphisms $\operatorname{diag}(C_1,\cdots,C_n)$ such that $C_1^{e_1}\cdots C_n^{e_n}=1$ for all $(e_1,\cdots,e_n)$ in Λ. A necessary and sufficient condition that a Lie algebra in $\mathcal{E}(V)$ be algebraic is that it contain $\mathfrak{g}(X)$ whenever it contains X.

The book contains basic results about algebraic groups and their Lie algebras, but makes no attempt to be complete in deducing analogues of theorems in the classical case of real Lie groups. The general principle which the author seems to have followed in this respect is to omit all arguments which are exactly the same as in the classical case.

Some of the misprints that were observed should be corrected as follows: p. 56, Def. 1: "s' est une specialisation de s" for "s est une specialisation de s'"; p. 59, l. 14b: "p^j" for "P_j"; p. 107, l. 9 and l. 11: "polynomes" for "rationelles"; p. 111, l. 16b: "$\mathcal{E}^R$" for "$\mathcal{E}\mathcal{R}$"; p. 119, l. 12: "$s(a)$" for "$s(b)$"; p. 118, l. 13: "$K(S)\subset K(v,t)$" for "$K(S)=K(v,t)$"; p. 121, l. 13: "cas b" for "cas a".

G. D. Mostow (Baltimore, Md.).

Følner, Erling. On two theorems of Pontrjagin. Comm. Sém. Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.] Tome Supplémentaire, 101–108 (1952).

The two theorems of Pontrjagin referred to in the title are those describing the structure of (1) separable, locally connected, compact abelian groups and (2) separable, connected, locally connected, locally compact abelian groups. In an earlier paper [Danske Vid. Selsk. Mat.-Fys. Medd. **25**, no. 19 (1950); these Rev. **11**, 640] the author gave a proof of the Pontrjagin duality theorem in the compact-discrete case based on his work with Bohr on closed modules in infinite-dimensional vector spaces [Studies and Essays Presented to R. Courant . . . , Interscience, New York, 1948, pp. 45–62; these Rev. **10**, 507]. In the present article the author uses the results of these two papers to give a proof of theorem (1) above and to simplify part of the argument in Pontrjagin's proof of theorem (2).

G. W. Mackey (Cambridge, Mass.).

Hu, Sze-tsen. Cohomology theory in topological groups. Michigan Math. J. **1**, 11–59 (1952).

Let Q be a topological group whose cohomology properties are to be studied; the abelian topological group G will be used as coefficient group. Q is supposed to operate upon G. The Eilenberg-MacLane definition covers the case of a discrete Q, whereas in essentially topological groups one must consider cohomology groups which are reduced with respect to the chains with empty support. Eventually, if

Q is a continuum, G a finite-dimensional vector group, and Q operates identically upon G, the group-theoretic definition of cohomology groups coincides with the topological definition. A third kind of cohomology groups are those with empty support; in the case of two dimensions they are related to the equivalence classes of cross-sections of a group extension of G by Q. The same notions are studied in local groups and applied to semi-simple groups. It is hardly possible to give briefly a more detailed account.

H. Freudenthal (Utrecht).

NUMBER THEORY

***Neiss, Fritz. Einführung in die Zahlentheorie.** S. Hirzel Verlag, Leipzig, 1952. viii+113 pp.

On order

Contents: Chap. 1. Divisibility properties of the integers. Chap. 2. Continued fractions. Chap. 3. Residue systems. Chap. 4. Arithmetic functions. Chap. 5. Special theorems. Chap. 6. Algebraic congruences. Chap. 7. Quadratic residues. Chap. 8. The quadratic reciprocity theorem. Chap. 9. Quadratic forms. Chap. 10. Problems.

In a little over 100 pages the author furnishes a very readable introduction to elementary number-theory. It may be remarked that he does not hesitate to use such terms as group, field, ring. Among the results included in the book we may in particular mention a proof of the two-square theorem for primes of the form $4m+1$ in Chapter 2 by means of continued fractions and a proof of the four-square theorem in Chapter 7. A particularly noteworthy feature of the book is the collection of 43 problems comprising Chapter 10. The author states that most of these problems had been collected by the late I. Schur. *L. Carlitz.*

Perron, Oskar. Bemerkungen über die Verteilung der quadratischen Reste. Math. Z. **56**, 122–130 (1952).

The distribution of quadratic residues and non-residues within a complete residue system is discussed. For example, in the case $p=4k-1$, if $r_1, r_2, \cdots, r_{2k}$ are the $2k$ quadratic residues (zero is considered as a residue) and a any integer prime to p, then the set $r_1+a, r_2+a, \cdots, r_{2k}+a$ always contains k residues and k non-residues. A similar result holds for the set of non-residues. Taking $a=1$ and the residue system $0, 1, \cdots, p-1$ this result shows that there are exactly k pairs of consecutive residues (including the pair 0, 1) and $k-1$ pairs of consecutive non-residues. In general, if the length of a pair b, c is defined to be the least positive integer congruent to either $b-c$ or $c-b$, then for any $a\leqq\frac{1}{2}(p-1)$ there are exactly k pairs of residues and $k-1$ pairs of non-residues of length a. Analogous results are obtained for the case $p=4k+1$. A schematic representation of the results for $p=4k-1$ is given. *W. H. Simons.*

Zassenhaus, Hans J. The quadratic law of reciprocity and the theory of Galois fields. Proc. Glasgow Math. Assoc. **1**, 64–71 (1952).

Using the notion of Gaussian sums in a finite field, the author gives another proof of the quadratic reciprocity law. This proof is a modernized version of Gauss's seventh proof.

W. H. Mills (New Haven, Conn.).

Ljunggren, Wilhelm. Eine elementare Auflösung der diophantischen Gleichung $x^3+1=2y^2$. Acta Math. Acad. Sci. Hungar. **3**, 99–101 (1952). (Russian summary)

The author's theorem [Skr. Norske Vid. Akad. Oslo. I. no. 9 (1942); these Rev. **6**, 169] that the only integral solutions of $x^3+1=2y^2$ have x values ±1 and 23 is proved now by elementary methods. *I. Niven* (Eugene, Ore.).

Buquet, A. Sur un critère d'indépendance de deux solutions données de l'équation diophantienne en nombres rationnels $x^3+dx+e=z^2$. Mathesis **61**, 183–193 (1952).

It was established by Mordell [Proc. Cambridge Philos. Soc. **21**, 179–192 (1922)] that the rational solutions of the equation of the title can be obtained rationally from a finite number of basic solutions. The writer calls two solutions independent if they cannot be obtained rationally from the same basic solution. If M_1 and M_2 are two rational points on the cubic, and if M_3 with coordinates (x_3, z_3) is the point on the cubic such that $M_1M_2M_3$ form a straight line, then it is well-known that M_3 is rational and the point $(x_3, -z_3)$ is called the resultant of M_1 and M_2. The tangential T of any rational point M on the curve is the rational point where the tangent at M cuts the curve. It is proved that if no point P exists on the curve having M_1, M_2, or their resultant as tangential, then M_1 and M_2 are independent. Various other geometric properties are obtained, and examples are given. *I. Niven* (Eugene, Ore.).

Leonardi, Raffaele. Sulla formazione dei sistemi di numeri equitotali. Boll. Un. Mat. Ital. (3) **7**, 345–350 (1952).

Various observations concerning systems of equations $\sum_{i=1}^{n}(x_i^a-y_i^a)=0$, $i=1, 2, \cdots, k$. *I. Niven.*

Gloden, A. Systèmes multigrades remarquables. Mathesis **61**, 278–280 (1952).

Drazin, M. P. A result concerning sequences of integers. Math. Gaz. **36**, 251–253 (1952).

Generalizing a previous result of Moser the author proves the following result: Let $s_1<s_2<\cdots$ be a sequence of integers. Assume that lim inf $s_n/n<2$. Then for every integer t there exist indices i and j so that $s_i-s_j=t$. Various generalizations are discussed. *P. Erdös.*

Gupta, Hansraj A table of values of $N_2(t)$. Res. Bull. East Panjab Univ. **1952**, no. 20, 13–93 (1952).

The function $N_2(t)$ is defined by $N_2(t)=\sum_{j\leqq t}n_2(j)$ in which $n_2(j)$ is the number of non-negative solutions (x, y) of (*) $x^2+y^2=j$, the solution (x, y) being considered as different from (y, x) in case $x\neq y$. The table gives for each possible $j<20000$ all solutions (x, y) of (*) in which $0\leqq x\leqq y$, together with the accumulated score $N_2(t)$. As a check, we have

$$N_2(t)=\sum[(t-j^2)^{\frac{1}{2}}]+[t^{\frac{1}{2}}]+1,$$

the sum extending over integers j for which $0\leqq j\leqq t^{\frac{1}{2}}$. The table was of use in preparing two previous papers [Proc. Indian Acad. Sci., Sect. A. **13**, 519–520 (1941); Proc. Nat. Inst. Sci. India **13**, 35–63 (1947); these Rev. **3**, 65; **8**, 566].

D. H. Lehmer (Los Angeles, Calif.).

Rodosskiĭ, K. A. On the theory of the ζ-function. Doklady Akad. Nauk SSSR (N.S.) **86**, 1069–1070 (1952). (Russian)

The author sketches a proof of the following result. Let T be a large integer; let

$$\tfrac{1}{2} \leqq \Delta \leqq 1-(\log\log T)^2/\log T, \quad \delta=(1-\Delta)/(2-3\Delta+2\Delta^2).$$

Then the number of rectangles $R^{(n)}$, $T+n-1\leqq t<T+n$, $\Delta\leqq\sigma\leqq 1$ $(n=1,\cdots,T)$, containing at least one point $\xi_0^{(n)}$ such that $|\zeta(\xi_0^{(n)})|\leqq(6T^{(2\Delta-1)\delta})^{-1}$ does not exceed $T^{(2\Delta+1)\delta}e^{c(\log\log T)^2}$. In the proof, there is used the approximate functional equation for the zeta function as well as methods used in a paper of Čudakov [Ann. of Math. (2) **48**, 515–545 (1947); these Rev. **9**, 11] and in a previous (unavailable) paper of the author [Ukrain. Mat. Žurnal **3**, 339–403 (1951)]. The author states that the result can be significantly improved when Δ is in the neighborhood of 1 by using estimates of exponential sums. *L. Schoenfeld.*

Sprague, Roland. Über additive Zerlegungen in lauter verschiedene Glieder einer Teilfolge der natürlichen Zahlenreihe. Math. Z. **56**, 258–260 (1952).

The author gives a modification of a theorem of Richert [Norsk Mat. Tidsskr. **31**, 120–122 (1949); these Rev. **11**, 646; see also references cited there] about partitioning numbers into distinct parts taken from a given set. The improvement lies in the reduction of the number of early special cases to examine. The author's theorem is as follows: Let M be an infinite set of strictly increasing integers $m_0=0<m_1<m_2<\cdots$ and let $N\geqq -1$ be an integer. Further suppose that there is an integer $k\geqq 0$ such that $2m_k-m_{k+1}\leqq N<m_{k+1}$ whereas $2m_r-m_{r+1}>N$ for all $r>k$. Then, if the numbers $N+1, N+2, \cdots, N+m_k$ may be partitioned into distinct parts taken from M, the same is true of all numbers exceeding N. With only 9 simple decompositions the author proves Richert's result that every number except 2, 5, 8, 12, 23, 33 is the sum of distinct triangular numbers. The method when applied to $m_k=k(k+1)(k+2)/6$ shows that every integer beyond 558 is a sum of distinct pyramidal numbers. *D. H. Lehmer* (Los Angeles, Calif.).

Prachar, K. Über einen Satz der additiven Zahlentheorie. Monatsh. Math. **56**, 101–104 (1952).

Suppose $k_0, k_1, k_2, \cdots$ is a strictly increasing sequence of non-negative integers, a is a fixed positive integer >1, $k(x)$ is the number of $a^{k_i}\leqq x$, and $R(x)$ is the number of distinct integers $\leqq x$ which can be expressed in form $p+a^{k_i}$, where p is a prime number. The author proves the existence of a positive constant c (depending only on a) such that $R(x)>cxk(x)/\log x$ for $x>2a^{k_0}$. The proof is a very straightforward adaptation of the method used by Romanoff for the special case $k_i=i$. Cf. Landau, Über einige neuere Fortschritte der additiven Zahlentheorie, Cambridge, 1937, Theorem 106. *P. T. Bateman* (Urbana, Ill.).

Šapiro-Pyateckiĭ, I. I. On a variant of the Waring-Goldbach problem. Mat. Sbornik N.S. **30**(72), 105–120 (1952). (Russian)

This paper is concerned with the number $J(N, r; C, \Delta)$ of solutions $(p_1, \cdots, p_r)$ of the inequality

$$|p_1^C+\cdots+p_r^C-N|<\Delta$$

where $C>1$ is not an integer and the p's are primes. Let $H(C)$ be the least value of r such that for each $\Delta>0$ there is an $N_0(\Delta)$ for which $J(N, r; C, \Delta)>0$ for all integers $N>N_0(\Delta)$. One result proved by the author is that $\limsup_{C\to\infty} H(C)/(C\log C)\leqq 4$; this may be compared with Vinogradov's [Trav. Inst. Math. Stekloff **23** (1947); these Rev. **10**, 599] result that $\limsup_{k\to\infty} G(k)/(k\log k)\leqq 3$ for the Waring problem where k is an integer and the primes p_m are replaced by positive integers x_m. Actually, a more general result is proved as are a number of more specialized results dealing with the cases $1<C<3/2$ and $r=2, 3$.

The results obtained considerably improve previous results of B. I. Segal [Ann. of Math. (2) **36**, 507–520 (1935)] who, however, dealt with positive integers x_m instead of primes p_m. Segal's results were based on an estimate of exponential sums due to van der Corput and hence were comparable to the older Hardy-Littlewood estimates of $G(k)$. The present author employs the Vinogradov estimates of exponential sums both for the case in which the summation letter runs over consecutive primes and for the case in which the summation letter runs over consecutive integers. In addition, Turán's [Izvestiya Akad. Nauk SSSR. Ser. Mat. **11**, 197–262 (1947); these Rev. **9**, 80] estimate for $\sum_p e^{i\xi p^C}$ is used. As in Segal's work an important part is played in a certain Fourier integral by the kernel $D(\Delta\xi)D^s(\delta\xi)$ where $D(x)=x^{-1}\sin x$ and ξ is the integration variable while Δ, δ are suitably chosen parameters. *L. Schoenfeld.*

Dénes, Peter. Beweis einer Vandiver'schen Vermutung bezüglich des zweiten Falles des letzten Fermat'schen Satzes. Acta Sci. Math. Szeged **14**, 197–202 (1952).

Let p be an odd prime, let $\Omega(\zeta)$ be the cyclotomic field of pth roots of unity, $\zeta=e^{2\pi i/p}$, $\lambda=1-\zeta$, $\Lambda=(1-\zeta)(1-\zeta^{-1})$, ϵ_0 a unit in $\Omega(\zeta)$. Vandiver [Trans. Amer. Math. Soc. **31**, 613–642 (1929)] proved the theorem: If (1) the second factor of the class-number of $\Omega(\zeta)$ is not divisible by p, and (2) none of the Bernoulli numbers B_{np} $(n=1, \cdots, (p-3)/2)$ is divisible by p^3, then the Fermat equation $\xi^p+\eta^p=\epsilon_0\Lambda^{mp}\psi^p$ has no non-zero solutions ξ, η, ψ in the real subfield $\Omega(\zeta+\zeta^{-1})$ of $\Omega(\zeta)$, which are relatively prime in pairs, and such that $\xi\eta\psi\equiv 0 \pmod p$ [this is the so-called second case]. He later conjectured [Proc. Nat. Acad. Sci. U. S. A. **16**, 298–304 (1930)] that there is no solution even when the restriction is removed that ξ, η, and ψ be relatively prime in pairs. This is the conjecture proved by the author. *R. Hull.*

Bergman, Gösta. A generalization of a theorem of Nagell. Ark. Mat. **2**, 299–305 (1952).

If A, B belong to a field Ω and satisfy the condition $4A^3-27B^2\neq 0$, then the equation $y^2=x^3-Ax-B$ defines a plane elliptic curve Γ, belonging to Ω, representable parametrically with Weierstrass equations:

$$x=\wp(u; 4A, 4B), \quad y=\tfrac{1}{2}\wp'(u; 4A, 4B).$$

The point (x, y) of Γ belongs to Ω if both x and y are in Ω; moreover, (x, y) is called an exceptional point of Γ, of order n (where n is a natural number) if nu is a period, while $n'u$ is not a period if $0<n'<n$.

In the present paper it is proved that, if Ω is an algebraic number field, of which A and B are integers, and if (x, y) belongs to Ω and is an exceptional point of Γ of order $n>1$, then x and y are integers in Ω in the following cases: i) if n is not a power of an odd prime; ii) if n is a power of 3 and 3 is not divisible by the 8th power of any prime ideal in Ω; iii) if n is a power of a prime $p>3$ and p is not divisible by the $(p-1)$th power of any prime ideal in Ω (the pth power does not suffice).

These results are obtained through the study of the algebraic equation in $x=\wp(u)$ given by $\sigma(nu)=0$. The case when $\Omega=k(1)$ had been considered first by T. Nagell [Skr. Norske

Vid.-Akad. Oslo I. **1935**, no. 1], who proved that x and y are then always ordinary integers. An incomplete extension of this result to quadratic and cubic fields had been obtained by G. Billing [Nova Acta Soc. Sci. Upsaliensis (4) **11**, no. 1 (1938)]. *B. Segre* (Rome).

Aigner, Alexander. Weitere Ergebnisse über $x^3+y^3=z^3$ in quadratischen Körpern. Monatsh. Math. **56**, 240–252 (1952).

The author first shows that any solution of the title equation in integers of a quadratic field $k(m^{\frac{1}{2}})$ can be reduced by permuting the variables and multiplying them by a common factor to the type in which x, y are conjugates and z is rational. This was previously regarded by Fueter as a specially simple particular case [S.-B. Heidelberger, Akad. Wiss. **4A**, no. 25 (1913)]. The equation with $|m|\equiv 1$ (3), $m<0$, was especially treated by Fueter (loc. cit.). Here the equation with $m\equiv 1$ (3), $m>0$, is proved insoluble if the class-number of $k((-3m)^{\frac{1}{2}})$ is prime to 3 (a similar criterion to Fueter's). The proof is by infinite descent and is elementary, except that it uses results of Scholz on the connection between the class-number of $k((-3m)^{\frac{1}{2}})$ and the class-number and fundamental unit of $k(m^{\frac{1}{2}})$ [J. Reine Angew. Math. **166**, 201–203 (1932)]. The method sheds light on the more difficult cases $|m|\equiv 2$ (3). The equation is also shown insoluble if m contains only prime factors of which 2 is a cubic non-residue. In particular, the title equation is insoluble in some fields with class-number divisible by 3, disproving a conjecture of Fueter. In conclusion there is a list of solutions in quadratic fields complementary to one of Fueter. *J. W. S. Cassels* (Cambridge, England).

Iseki, Kanesiroo. On the imaginary quadratic fields of class-number one or two. Jap. J. Math. **21** (1951), 145–162 (1952).

Let $-\Delta$ be the discriminant of an imaginary quadratic field and h the class number. It is well known that there is at most one such field with $\Delta>163$ and $h=1$. In this paper it is shown that there is at most one such field with $\Delta>90,000$ and $h\leqq 2$. In an earlier paper [Nat. Sci. Rep. Ochanomizu Univ. **3**, 23–29 (1952); these Rev. **14**, 140] it was shown that there are exactly 18 values of Δ less than 6000 with $h=2$, the largest being 427. Given any integer h_0, the methods of the present paper can be used to calculate a number $N(h_0)$ explicitly, such that there is at most one imaginary quadratic field with $\Delta>N(h_0)$ and $h\leqq h_0$. (See the following review.) The method is based on a comparison of the zeros of the two functions $L(s,\chi_1)$ and $F(s)=\zeta(s)L(s,\chi_1)L(s,\chi_2)L(s,\chi_1\chi_2)$, where $L(s,\chi_i)$ is the L-series corresponding to the character χ_i given by the Kronecker symbol $(-\Delta_i/n)$. If $\Delta_1>90,000$ and the corresponding class number h_1 is 1 or 2, then $L(s,\chi_1)$ has a zero between $1-5\Delta_1^{-1/2}$ and $1-h_1\Delta_1^{-1/2}$. On the other hand, if $\Delta_2>\Delta_1>90,000$ and h_1, $h_2\leqq 2$, then a detailed investigation reveals that $F(s)$ is always positive in this interval. *W. H. Mills* (New Haven, Conn.).

Tatuzawa, Tikao. On a theorem of Siegel. Jap. J. Math. **21** (1951), 163–178 (1952).

Let k be a positive integer. Let χ be a non-principal primitive real character modulo k, and let $L(s,\chi)$ be the corresponding L-series. Siegel has shown that $L(1,\chi)>c(\epsilon)k^{-\epsilon}$ for any positive ϵ, where $c(\epsilon)$ is a positive constant that depends only on ϵ but cannot be given explicitly [Acta Arith. **1**, 83–86 (1935)]. Here it is shown that $L(1,\chi)>0.1\epsilon k^{-\epsilon}$, with at most one exception. Furthermore, if $\frac{1}{2}>\epsilon>0$ and $k\geqq \max(e^{1/\epsilon}, e^{11.2})$, then $L(1,\chi)>0.655\epsilon k^{-\epsilon}$ with at most one exception. From the latter result it is simple to deduce the following theorem: Given any positive integer h_0 there is at most one imaginary quadratic field with class number $h\leqq h_0$ and discriminant $-k$, $k\geqq 2100\, h_0^2 \log^2 (13\, h_0)$. As in the special case $h_0=2$ treated by Iseki, the proof is based on a comparison of the zeros of $L(s,\chi_1)$ and $F(s)$ [see the preceding review]. The actual method by which this comparison is made is considerably different in the two papers. *W. H. Mills* (New Haven, Conn.).

Weil, André. Jacobi sums as "Grössencharaktere". Trans. Amer. Math. Soc. **73**, 487–495 (1952).

Let k be an algebraic number-field of degree d and let $\mathfrak{m}$ be an (integral) ideal of k. Following Hecke, a complex-valued function $f(\mathfrak{a})$ defined and $\neq 0$ for all (integral) ideals $\mathfrak{a}$ prime to $\mathfrak{m}$ is a Grössencharakter provided $f(\mathfrak{ab})=f(\mathfrak{a})f(\mathfrak{b})$ if there are rational integers e_λ and complex numbers c_λ, $1\leqq\lambda\leqq d$, such that if α is an integer of k, $\alpha\equiv 1$ ($\mathfrak{m}$), and $\alpha_1, \cdots, \alpha_d$ are the conjugates of α, then $f((\alpha))=\prod \alpha_\lambda^{e_\lambda}|\alpha_\lambda|^{c_\lambda}$; $\mathfrak{m}$ is called a defining ideal for f. In the next place let $m>1$, and ζ be a primitive mth root of unity over Q, the field of rationals. Let $\mathfrak{p}$ be any prime ideal in $Q(\zeta)$ prime to m and define the multiplicative character $\chi_\mathfrak{p}(x)\equiv x^{(q-1)/m}(\mathfrak{p})$, where $q=N\mathfrak{p}$. Now let $r\geqq 1$ and let $a=(a_1, \cdots, a_r)$ be a set of integers (mod m) and put $J_a(\mathfrak{p})=(-1)^{r+1}\sum \chi_\mathfrak{p}^{a_1}(x_1)\cdots\chi_\mathfrak{p}^{a_r}(x_r)$, the summation extending over all $x_1, \cdots, x_r$ ($\mathfrak{p}$) such that $x_1+\cdots+x_r\equiv -1$ ($\mathfrak{p}$); also let $J_a(\mathfrak{ab})=J_a(\mathfrak{a})J_a(\mathfrak{b})$ for all $\mathfrak{a}$, $\mathfrak{b}$ prime to m. The main result of the paper is contained in the theorem: For each $a\not\equiv(0)$, the function $J_a(\mathfrak{a})$ is a Grössencharakter and m^2 is a defining ideal for it.

As an application of this theorem a proof is given of the following conjecture of Hasse. Consider the curve (*) $Y^e=\gamma X^f+\delta$, where $2\leqq e\leqq f$ and γ, δ are non-zero elements of an algebraic number-field k of characteristic prime to ef. If $\mathfrak{p}$ is a prime ideal of k, prime to $ef\gamma\delta$, then (*) reduced mod $\mathfrak{p}$ defines a curve over the finite field $GF(q)$, $q=N\mathfrak{p}$. Let $Z_\mathfrak{p}(u)$ denote the zeta-function of that curve and define $Z(s)=\prod_\mathfrak{p} Z_\mathfrak{p}(N\mathfrak{p}^{-s})$; then Hasse conjectured that $Z(s)$ is a meromorphic function that satisfies a functional equation of the usual type. *L. Carlitz* (Durham, N. C.).

Satake, Ichiro. On a generalization of Hilbert's theory of ramification. Sci. Papers Coll. Gen. Ed. Univ. Tokyo **2**, 25–39 (1952).

The author begins by reviewing the known ramification theory in an infinite normal extension over a ground field with discrete valuation and perfect residue class field. He calls such an extension H_1 if the Hasse function $\varphi(u)$ is finite for all u, $0\leqq u<\infty$. He then shows that the whole theory, including the Herbrand theorem, can be generalized to certain extensions Ω/K in which the valuation on K is not necessarily discrete; for example, in case $k\subset K\subset\Omega\subset\Omega^*$ and Ω^*/k is H_1. Other partial results are given and difficulties of further generalizations are mentioned. In a final section devoted to arithmetic, the author observes that an abelian extension of a number field or $\mathfrak{p}$-adic field is H_1, and proves the Hasse theorem on the norms of unit groups by considering their images under the reciprocity law isomorphism. *J. T. Tate* (Princeton, N. J.).

Moriya, Mikao. Eine notwendige Bedingung für die Gültigkeit der Klassenkörpertheorie im Kleinen. Math. J. Okayama Univ. **2**, 13–20 (1952).

The author resumes the work of the reviewer on the implications of certain theorems of local class field theory (theorems are taken as axioms for a field which is complete

with respect to a discrete rank one valuation) on the algebraic structure of the residue class field (see also a paper of the reviewer [Amer. J. Math. **60**, 75–100 (1938)]). It is shown how (i) the "limitation theorem" (incidentally a weak form suffices), (ii) the "isomorphism theorem", and (iii) the "inversion theorem" imply that the residue class field k is perfect and admits for each integer n precisely one cyclic extension of degree n. Assumption (i) implies that k is perfect; (i) and (ii) imply that all extensions of k are cyclic (using the theory of inertial extensions); and finally, adding (iii), the existence of an extension of degree n over k for each given n is established. *O. F. G. Schilling.*

Nakayama, Tadasi. On a 3-cohomology class in class field theory and the relationship of algebra- and idèle-classes. Ann. of Math. (2) **57**, 1–14 (1953).

The main result of this paper and a sketch of a proof are already contained in a previous publication of the author [Proc. Japan Acad. **27**, 401–403 (1951); these Rev. **13**, 916]. Let k be an algebraic number field, K/k a finite normal extension with Galois group $\mathfrak{f}$, K^* the multiplicative group of K, C_K its idèle class group. The class field theory provides a canonical generator $u_{K/k}$ for the 2-dimensional cohomology group $H^2(\mathfrak{f}, C_K)$. [See Weil, J. Math. Soc. Japan **3**, 1–35 (1951); these Rev. **13**, 439, and Nakayama, Ann. of Math. (2) **55**, 73–84 (1952), pp. 78–84; these Rev. **13**, 629]. The present result is a determination of those $\mathfrak{f}$-normal algebra classes over K whose associated Teichmüller cohomology class is the natural image of $u_{K/k}$ in $H^3(\mathfrak{f}, K^*)$. For a precise statement of the result, for the terminology, and for further references, see the review of the author's previous paper cited above. In the proof given here, the author concentrates on a detailed analysis of the constructions of Teichmüller cocycles and of idèle class cocycles, exhibiting a certain parallelism between them. That such an analysis is not needed for the final result was observed rather late by the reviewer. The direct procedure is indicated in a note added in proof at the end of the paper.

It has been shown by Artin and Tate that the natural homomorphism $H^2(\mathfrak{f}, C_K) \to H^3(\mathfrak{f}, K^*)$ is onto. (This follows at once from Th. 2 in J. Tate, ibid. **56**, 294–297 (1952); these Rev. **14**, 252). Consequently, the present result provides a description, in terms of algebra classes, of the canonical generator for the whole group $H^3(\mathfrak{f}, K^*)$. (The symbol at the end of the last line but one of the paper under review should be P_K).

The author gives an interpretation of his result involving the group $I_K \times_{K^*} A^*$, where A^* is the multiplicative group of the invertible elements of a $\mathfrak{f}$-normal simple algebra A with center K, and where the subscript K^* indicates that the common subgroup K^* of A^* and I_K (the idèle group of K) is amalgamated. Using the $\mathfrak{f}$-normality of A, one can define on this group the structure of a $\mathfrak{f}$-kernel, in the sense of Eilenberg-MacLane. Since the Teichmüller class of A splits in I_K, this kernel is "extendible", i.e., there exists a group extension of $I_K \times_{K^*} A^*$ by $\mathfrak{f}$ which induces the given kernel structure. The main result gives a criterion in terms of the invariants of A for the existence of such a group extension which has the further property that its class of factor systems maps into the canonical generator of $H^2(\mathfrak{f}, C_K)$ by a natural projection.

Reviewer's note: The arithmetic significance of the canonical generator of $H^3(\mathfrak{f}, K^*)$ becomes very striking in an as yet unpublished result of J. Tate: the cup multiplication by this generator maps the integral 2-dimensional homology group of $\mathfrak{f}$ (which depends only on $\mathfrak{f}$) homomorphically onto the local-global norm residue group $[k^* \cap N_{K/k}(I_K)]/N_{K/k}(K^*)$, whence one has a highly interesting generalization of Hasse's norm theorem.

G. Hochschild (Urbana, Ill.).

Ramanathan, K. G. Units of quadratic forms. Ann. of Math. (2) **56**, 1–10 (1952).

The notation and terminology employed are those of the author's earlier paper [Amer. J. Math. **73**, 233–255 (1951); these Rev. **13**, 628], and two papers of Siegel [(1) Ann. of Math. **44**, 674–689 (1943); (2) ibid. **45**, 577–622 (1944); these Rev. **5**, 228; **6**, 38]. If S is the matrix of a quadratic form with coefficients integral in an algebraic number field K, its group of units (automorphs) is $\Gamma(S)$, consisting of the matrices U, integral in K, such that $U'SU=S$. By way of preparation, it is shown that all compact subgroups of $G(m, c)$, the complex orthogonal group of order m, that is, of complex m-by-m matrices C such that $C'C=E$, are conjugates of $G(m, R)$, the real orthogonal group. Hence the coset space $G(m, c)/G(m, R)$ is homeomorphic to the algebraic manifold H of positive hermitean matrices Y which are orthogonal. Further, H is parametrized by $m(m-1)/2$ real variables whose domain is the bounded region: $E-\bar{Y}^2>0$, on which $G(m, c)$ acts transitively. The euclidean volume of H is finite, but its volume for the invariant element $dV=|E-\bar{Y}^2|^{(1-m)/2}\{dY\}$ is infinite. Similar preparatory results are obtained for the group $G(n, m-n)$ of real matrices C such that $C'DC=D$, where

$$D=\operatorname{diag}(1, \cdots, 1, -1, \cdots, -1),$$

n 1's and $m-n$ (-1)'s. Then with S is associated an algebraic manifold constructed out of the topological product of r_1 groups like $G(n, m-n)$, and r_2 groups like $G(m, c)$, as is H above from $G(m, c)$, where r_1 is the number of real conjugates of K, $2r_2$ the number of complex conjugates. This manifold reduces to a point if $r_2=0$ and S is totally definite. Some results are that $\Gamma(S)$ is finite if and only if $r_2=0$ and S is totally definite, or if $r_2=0$ and S is binary decomposable; in any case, $\Gamma(S)$ is finitely generated. Other results are obtained, extending those of Siegel for the rational case, relating to the properties of $\Gamma(S)$ from the standpoint of discontinuous groups [Siegel (1)], furnishing examples of discontinuous groups of the first kind with normal fundamental sets. *R. Hull* (Lafayette, Ind.).

Brandt, Heinrich. Über das Mass positiver ternärer quadratischer Formen. Math. Nachr. **6**, 315–318 (1952).

The weight (Mass) of a genus G of positive definite ternary quadratic forms is, by Eisenstein's definition [J. Reine Angew. Math. **35**, 117–136 (1847)], the number $M=\sum 1/t$, summed over the classes in G, where t is the number of proper (i.e., of determinant 1) integral automorphs of a form in the class. Eisenstein gave without proof a formula for M in the case of properly primitive forms of odd determinant. Smith [Collected mathematical papers, v. I, Oxford, 1894, pp. 455–509] supplied a proof and also gave the corresponding formulas for all properly and improperly primitive cases. The formulas [loc. cit., p. 499] involve the invariants Ω and Δ of a form f of G, character-values of f and its adjoint, and a numerical factor ζ with values tabulated for 20 cases: type A (16 cases), types B and C (2 cases each). Brandt here considers ternary forms with odd as well as even coefficients of the product terms, that is, in terms of forms, he studies properly primitive ones and the halves of improperly primitive ones. He gives a formula for M which is simpler than Smith's in that ζ is

replaced by a factor κ for which there are only 3 cases, viz., the "types" A, B, C, instead of 20. He does not derive his formulas directly by the usual transcendental means but verifies them using Smith's formulas. The relation between his formulas and Smith's is thus made clear. [The same considerations have advantages in the character-theory of ternary forms which the author has developed in the paper reviewed below.] *R. Hull* (Lafayette, Ind.).

Brandt, H. Zur Zahlentheorie der ternären quadratischen Formen. Math. Ann. **124**, 334–342 (1952).

The ternary quadratic form

$$f=a_1x_1^2+a_2x_2^2+a_3x_3^2+a_4x_2x_3+a_5x_3x_1+a_6x_1x_2,$$

with rational integral a's, may be written $f=tf_0$, where $t=\pm$(g. c. d. of the a's) and f has positive signature. With sign so chosen, t is called the coefficient-divisor of f and f is called primitive when t is $+1$. Write x for the row-matrix (x_1, x_2, x_3), or the set of variables, and let A be the symmetric 3-by-3 matrix of the a's such that $f=xAx'/2$. The elements of A have g. c. d. $2|t|$ or $|t|$, according as a_4, a_5, a_6 are all even or not, and f is called of the second or first kind in the respective cases. The discriminant of f is defined to be $-\frac{1}{2}|A|$. Let F be the primitive adjoint of a primitive f and hence of the same signature. The identities

$$(f_1y_1+f_2y_2+f_3y_3)^2-4f(x)f(y)=I_1F(s),\quad f_i=\partial f(x)/\partial x_i,$$
$$(F_1y_1+F_2y_2+F_3y_3)^2-4F(x)F(y)=I_2f(s),\quad F_i=\partial F(x)/\partial x_i,$$

where $s=(x_2y_3-x_3y_2,\ x_3y_1-x_1y_3,\ x_1y_2-x_2y_1)$, define the first and second invariants I_1 and I_2 of f, which are the second and first invariants of F. Also $I_1^2I_2=16d$ and $I_2^2I_1=16D$, and I_1, I_2, d, and D are all negative when f is positive definite and all positive when f is indefinite (of positive signature). Ternary forms fall into three types: (1) I_1 odd $16|I_2$; (2) $4|I_1$, $4|I_2$; (3) I_2 odd, $16|I_1$. Ternary prime-discriminants are $-4, 8, -8, -3, 5, -7, -11, 13, \cdots$. The paper is devoted chiefly to showing that all characters of f belong to the prime-discriminants into which I_1 factors, those of F similarly from I_2, provided that when I_1 and I_2 are not both divisible by 16, certain signs are taken into consideration. The claims of the author for the advantages of his notations and methods over those of Smith [Collected mathematical papers, v. I, Oxford, 1894, pp. 455–509] for Gaussian forms (a_4, a_5, a_6 all even) appear to be justified. [See also the preceding review.] *R. Hull.*

★Chabauty, Claude. Sur des problèmes de géométrie des nombres. Algèbre et Théorie des Nombres. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 24, pp. 27–28. Centre National de la Recherche Scientifique, Paris, 1950.

Cassels, J. W. S. On a paper of Niven and Zuckerman. Pacific J. Math. **2**, 555–557 (1952).

Let $0\leqq a_i<10$, a_i integer. Denote by

$$R_m(a_1, a_2, \cdots, a_r; b_1, b_2, \cdots, b_s)$$

the number of solutions of

$$b_n=a_1,\ b_{n+1}=a_2,\ \cdots,\ b_{n+r-1}=a_r,$$
$$0<n<n+r\leqq s,\ n\equiv m \pmod{r}.$$

Let $x_1, x_2, \cdots$ be an infinite sequence of integers $0\leqq x_i<10$. Niven and Zuckerman showed [same J. **1**, 103–109 (1951); these Rev. **13**, 438] that if

$$(1)\quad \lim_{N\to\infty}\frac{1}{N}\sum_{m=1}^{r}R_m(a_1, a_2, \cdots, a_r; x_1, x_2, \cdots, x_N)=10^{-r}$$

for all r and integers $a_1, a_2, \cdots, a_r$ $(0\leqq a_i<10)$ (i.e., if $x_1, x_2, \cdots$ is normal), then

$$(2)\quad \lim_{N\to\infty}\frac{1}{N}R_m(a_1, a_2, \cdots, a_r; x_1, \cdots, x_N)=r^{-1}10^{-r}$$

for all integers r, m, and a_i. The author gives a very simple and ingenious proof of this result. *P. Erdös.*

Macon, Nathaniel. Some theorems on the approximation of irrational numbers by the convergents of their continued fractions. J. Elisha Mitchell Sci. Soc. **67**, 99–107 (1951).

Let ξ be a positive irrational number, let the convergents of the continued fraction expansion of ξ be A_n/B_n, and put $|\xi-A_n/B_n|=(\lambda_nB_n^2)^{-1}$. It is shown that for every $n\geqq0$, the inequalities $\lambda_n\lambda_{n+1}\lambda_{n+2}>27/4$ and $\lambda_n\lambda_{n+1}\lambda_{n+2}\lambda_{n+3}>625/36$ hold, these being best possible constants. Moreover, at least $m+1$ of any $3m+2$ consecutive λ_i exceed $\sqrt{5}$, or at least m of them exceed 3. This paper is closely related to earlier work of A. Brauer and the author [Amer. J. Math. **71**, 349–361 (1949); **72**, 419–424 (1950); these Rev. **10**, 513; **11**, 501].
W. J. LeVeque (Ann Arbor, Mich.).

Kurzweil, Jaroslav. A contribution to the metric theory of diophantine approximations. Czechoslovak Math. J. **1**(76) (1951), 149–178 (1952) = Čechoslovack. Mat. Ž. **1**(76) (1951), 173–203 (1952).

If $g(q)$ is a given positive function defined for real $q>0$, an irrational number is said to admit the approximation $g(q)$ if there exist infinitely many pairs of integers p, q $(q>0)$ such that

$$\left|x-\frac{p}{q}\right|<\frac{1}{q^2g(q)}.$$

Khintchine showed that the Lebesgue measure of the set of x in $(0, 1)$ which admit the approximation $g(q)$ is zero if $\int^\infty dx/\{xg(x)\}$ converges and unity if this integral diverges. In the case of convergence Jarník [Mat. Sbornik **36**, 371–382 (1929); Math. Z. **33**, 505–543 (1931)] investigated the set of x in $(0, 1)$ which admit the approximation $g(q)$ by means of Hausdorff measure. Here the author considers the corresponding problem of the set Q_g of x in $(0, 1)$ which do not admit the approximation $g(q)$ when the integral diverges. The functions $f_1(d)$ and $f_2(d)$ are defined by

$$f_1(d)=\exp\left\{\frac{2}{3}\int_w^{d^{-1}}\frac{dx}{xg(x)}\right\},\quad f_2(d)=\exp\left\{2\int_w^{d^{-1}}\frac{dx}{xg(x)}\right\}$$

and $g(x)$ is assumed to satisfy certain additional subsidiary conditions. The main result proved is the theorem that if $g(q)>1000$, then the Hausdorff measure of Q_g formed with respect to f_1 is zero whilst that formed with respect to f_2 is infinity. To prove this the numbers in $(0, 1)$ are expressed as continued fractions and divided into intervals each of which consists of numbers whose expansions as a continued fraction begin in a prescribed way. These intervals are used to cover sets associated with Q_g but the details of the analysis are too complicated to be described here. The theorem is used to obtain the Hausdorff dimension of Q_g for functions $g(q)$ of the form $\log^{\alpha_1} q\,\log_2^{\alpha_2} q\cdots\log_n^{\alpha_n} q$, where the suffix denotes a repeated logarithm and the Hausdorff measure is taken with respect to functions of the form

$$f_e(d)=\exp\left\{\log^{-e_1}\frac{1}{d}\,\log_2^{-e_2}\frac{1}{d}\cdots\log_r^{-e_r}\frac{1}{d}\right\}.$$

In particular, if $g(q)=\log^{\alpha} q$ $(0<\alpha\leqq 1)$ and

$$f_s(d)=\exp(\log^{-s} 1/d) \quad (-1<s\leqq 0)$$

then $\dim Q_g=\alpha-1$. The case g a constant greater than 1000 is also considered and it is shown that, if $f_s(d)=d^{s-1}$ $(0<s\leqq 1)$, then $1-0.99_s^{-1}\leqq\dim Q_g\leqq 1-0.25_s^{-1}$.

R. A. Rankin (Birmingham).

ANALYSIS

★Dynkin, E. B., i Uspenskiĭ, V. A. Matematičeskie besedy. Zadači o mnogocvetnoĭ raskraske. Zadači iz teorii čisel. Slučaĭnye bluždaniya. [Mathematical conversations. Map coloring problems. Problems from number theory. Random walks.] Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, Leningrad, 1952. 288 pp. 5.65 rubles.

This book consists of 213 problems and their detailed solutions, somewhat in the style of Pólya and Szegö [Aufgaben und Lehrsätze aus der Analysis, Springer, Berlin, 1925], but on a more elementary and picturesque level, and of course with a different selection of material, as indicated by the title. In the first part, maps of 2, 3, 4, and 5 colors are discussed, including the theorems of Volynskiĭ, of Euler, and of Lusternik and Schnirelmann. The second part discusses congruences, m-adic and p-adic numbers, Fibonacci and Pascal sequences, and a Pell's equation. The third part discusses random walks and Markov chains with a finite or denumerable number of states.

K. L. Chung.

Hoffman, Alan J. On approximate solutions of systems of linear inequalities. J. Research Nat. Bur. Standards **49**, 263–265 (1952).

Let $Ax\leqq b$ be a consistent system of linear inequalities. The principal result is a quantitative formulation of the fact that if x "almost" satisfies the inequalities, then x is "close" to a solution. It is further shown how it is possible in certain cases to estimate the size of the vector joining x to the nearest solution from the magnitude of the positive coordinates of $Ax-b$. (Author's summary.)

R. Bellman (Santa Monica, Calif.).

Pizzetti, Ernesto. Dalle proporzioni continue alle progressioni. Metron **16**, nos. 3–4, 27–39 (1952).

Allen, A. C. A generalization of a theorem by Hardy and Littlewood. Proc. Amer. Math. Soc. **3**, 727–731 (1952).

Let $\Phi(x)$ be non-negative and integrable in $\langle 0, 1\rangle$, and let $\bar{\Phi}(x)$ be the rearrangement of Φ in decreasing order. The following theorem generalises a result of Hardy and Littlewood [Acta Math. **54**, 81–116 (1930)] in the special case $k\equiv 1$. Let $f(x)$ and $k(x)$ be non-negative, $k(x)$ decreasing, and $f(x)$, $k(x)$, and $k(x)\bar{f}(x)$ be integrable in $\langle 0, 1\rangle$. (i) If $\Theta(x)=\max_{0<h\leqq x} h^{-1}\int_0^h k(t/n)f(x-h+t)dt$, then

$$\overline{\Theta}(x)\leqq\frac{1}{x}\int_0^x k\left(\frac{t}{x}\right)\bar{f}(t)dt \qquad \text{for} \quad 0<x\leqq 1.$$

(ii) If $s(y)$ increases for $y\geqq 0$, then

$$\int_0^1 s(\Theta(x))dx\leqq\int_0^1 s\left\{\frac{1}{x}\int_0^1 f(t)dt\right\}dx.$$

(i) and (ii) are equivalent results.

W. W. Rogosinski.

Calculus

Mersman, W. A. Evaluation of an integral occurring in servomechanism theory. Pacific J. Math. **2**, 627–632 (1952).

The theory of the best design of linear servomechanisms, as described by R. S. Phillips [see James, Nichols, and Phillips, Theory of servomechanisms, McGraw-Hill, New York, 1947; these Rev. **11**, 517] leads especially to the evaluation of the integral

$$I=\frac{1}{2\pi i}\cdot\int_{-\infty i}^{\infty i}\frac{g(x)dx}{h(x)\cdot h(-x)}$$

with $g(x)=\sum_{k=1}^{n} g_k x^{2(n-k)}$ and $h(x)=\sum_{k=0}^{n} a_k x^{n-k}$, $a_0\neq 0$, a_k real. A procedure for evaluating the integral I and formulae for I with regard to special polynomials g and h of low degree have been given by Phillips. The author develops a general formula, representing I as a rational function of the coefficients a_k and g_m. The result is

$$I=\frac{(-1)^{n+1}\cdot\|g_{ij}\|}{2a_0\cdot\|c_{ij}\|}$$

with $c_{ij}=a_{2i-j}$, $g_{ij}=g_i$ for $j=1$ and $g_{ij}=c_{ij}$ for $j>1$. It is to be understood that $a_r=0$ for $r<0$ and for $r>n$.

H. Bückner (Berlin).

Fettis, Henry E. An integral in the theory of wave-guide-fed slots. J. Appl. Phys. **23**, 1409 (1952).

The author gives an explicit expression for the integral

$$\int_0^{\pi/2}(1-k^2\sin^2\theta)^{-1}[1+\cos(\pi k\sin\theta)]d\theta$$

in terms of

$$\int_0^{\pi} J_0(kt)\begin{matrix}\cos\\ \sin\end{matrix}\, tdt.$$

A. Erdélyi (Pasadena, Calif.).

Corinaldesi, E., and Trainor, L. Evaluation of integrals in the theory of atomic scattering of electrons. Nuovo Cimento (9) **9**, 940–945 (1952).

The authors develop methods for the evaluation in closed form of certain Coulomb and exchange integrals, and give the values of the integrals for $1s-1s$, $1s-2s$, and $1s-2p$ transition.

A. Erdélyi (Pasadena, Calif.).

Theory of Sets, Theory of Functions of Real Variables

Nakahara, Isamu. Sur la classe projective d'un ensemble défini par l'induction transfinie. Proc. Japan Acad. **28**, 336–338 (1952).

It is proved (Th. 2) that the Lebesgue set Z is an elementary set relative to $C_0\times C\times\mathfrak{J}$, C and $\mathfrak{J}$ being a Cantor set and the set of irrational numbers respectively; C_0 is the set of all M_x $(x\in C)$ such that M_x is well ordered [notation as

in Kuratowski, Fund. Math. **27**, 269–276 (1936), where it is proved that Z is a projective set of class $\leqq 3$; according to von Neumann Z is a difference of A-sets, and thus of class $\leqq 2$ [cf. Kuratowski and von Neumann, Ann. of Math. (2) **38**, 521–525 (1937)]. *Đ. Kurepa* (Zagreb).

Sierpiński, Wacław. Sur quelques propositions qui entraînent l'existence des ensembles non mesurables. Soc. Sci. Lett. Varsovie. C. R. Cl. III. Sci. Math. Phys. **42** (1949), 36–40 (1952). (Polish summary)

Consider the following propositions (n denotes a natural number): (Z_n) The axiom of choice is true for sets consisting of n elements. (T) There are not more than $2^{\aleph_0}$ distinct linear sets which can be brought into coincidence with the set of rational numbers by a translation. (L) There exists a linear set which is not Lebesgue measurable. It is shown, without using the axiom of choice, that (T) implies (L), and that, if $n>1$, (Z_n) implies (L). The author proved earlier [Fund. Math. **10**, 177–179 (1927)] (without the help of the axiom of choice) that (L) follows from (Z_2). *F. Bagemihl* (Rochester, N. Y.).

Nožička, František. The theorem on the supremum and theorems equivalent to it. Časopis Pěst. Mat. **76**, 121–140 (1951). (Czech)

This is an expository article, proving the equivalence of the following assertions for the real number system: (I) bounded sets have least upper bounds; (II) monotone sequences converge if and only if they are bounded; (III) Cauchy sequences converge; (IV) bounded infinite sets admit points of accumulation. All 12 of the possible implications are painstakingly established. *E. Hewitt.*

Kozlova, Z. I. Mutual relations among theorems of multiple separability. Izvestiya Akad. Nauk SSSR. Ser. Mat. **16**, 389–404 (1952). (Russian)

Let X be a set, (U) a family of subsets of X, (CU) the family of all complements of sets in (U). Let

$$M(U)=(U)\cap(CU).$$

One may ask, given sets $E_1, E_2 \in (U)$ such that $E_1\cap E_2=0$, if there exist sets $H_i\supset E_i$ such that H_i are in some other family of sets, e.g., $M(U)$ $(i=1, 2)$ and such that $H_1\cap H_2=0$. This is the problem of simple separability. The same question may be raised for sets $E_1, \cdots, E_k \in (U)$ such that $\bigcap_{i=1}^k E_i=0$. For infinite sequences $E_1, E_2, \cdots, E_n, \cdots \in (U)$, one may ask analogous questions for the cases $\bigcap_{i=1}^\infty E_i=0$, $\limsup E_i=0$, $\liminf E_i=0$, and other operations on E_i, $i=1, \cdots, \infty$. The author lists 27 possible axioms of separability, and in 11 theorems explores a number of the implications that hold regarding them. *E. Hewitt.*

Bertolini, Fernando. Insieme limite degli aggregati gruppali d'insiemi. Portugaliae Math. **11**, 119–128 (1952).

Some elementary remarks on convergence of filter-bases in compact spaces. *V. L. Klee, Jr.*

Dubrovskiĭ, V. M. On a property of a formula of Nikodým. Doklady Akad. Nauk SSSR (N.S.) **85**, 693–696 (1952). (Russian)

The author completes some results given in his former papers [same Doklady (N.S.) **58**, 737–740 (1947); Mat. Sbornik N.S. **20**(62), 317–329 (1947); these Rev. **9**, 275, 19]. Let $\mathfrak{M}$ be a family of subsets e of an abstract set $\mathfrak{A}$, closed under the formation of complements and of countable unions, A a set of indices α, $\Phi(\alpha, e)$ a family of denumerably additive real-valued set-functions on $\mathfrak{M}$, $\mathfrak{F}(\alpha, e)$ their total variations, $M(e)\geqq 0$ a denumerably additive measure on $\mathfrak{M}$ such that $M(e)=0$ implies $\Phi(\alpha, e)=0$ for all α ("basis" for $\Phi(\alpha, e)$). Let $f(\alpha, x)$ be a family of M-summable real-valued functions of the point x varying in $\mathfrak{A}$. Given $\Phi(\alpha, e)$ and an infinite sequence $S=(e_1, e_2, \cdots, e_n, \cdots)$ of disjoint sets $e_n \in \mathfrak{M}$, put

$$\mathfrak{d}(S)=\lim_{n\to\infty}[\sup_{\alpha\in A}\mathfrak{F}(\alpha, e_{n+1}+e_{n+2}+\cdots)]\leqq+\infty.$$

$\mathfrak{D}=\sup_S \mathfrak{d}(S)$ is called the index of countable additivity of the family $\Phi(\alpha, e)$. Given $f(\alpha, x)$, $N>0$, and $M(e)$, put

$$E(\alpha, N)=\{x \mid x\in\mathfrak{A}, |f(\alpha, x)|>N\},$$

$$D(\alpha, N)=\int_{E(\alpha, N)}(|f(\alpha, x)|-N)d_xM.$$

$\Delta=\lim_{N\to\infty}[\sup_{\alpha\in A} D(\alpha, N)]$ is called the index of M-summability of the family $f(\alpha, x)$. The family $\Phi(\alpha, e)$ is said to enjoy the property of uniform countable additivity if for every S the sequence $\Phi(\alpha, e_{n+1}+e_{n+2}+\cdots)$ tends to 0 uniformly for all $\alpha \in A$. A family $f(\alpha, x)$ of M-summable functions is said to enjoy the property of equal M-summability if $D(\alpha, N)$ tends to 0 with $N\to\infty$ uniformly for all $\alpha\in A$.

The following theorems are proved. (1) $\Phi(\alpha, e)$ enjoys the property of uniform countable additivity if and only if $\mathfrak{D}=0$. (2) $f(\alpha, x)$ enjoys the property of equal M-summability if and only if $\Delta=0$. (3) Let $f(\alpha, x)$ and $\Phi(\alpha, e)$ be related by the formula $\Phi(\alpha, e)=\int_e f(\alpha, x)d_xM$ (for given M); if $\Phi(\alpha, e)$ is uniformly bounded, then the index of countable additivity of Φ is equal to the index of M-summability of f. [Remark of the reviewer: It suffices to suppose that $\Phi(\alpha, e)$ is bounded for each e separately, because the uniform boundedness follows; see the paper of the reviewer, Monatsh. Math. Phys. **40**, 418–426 (1933).] The author shows that if the condition of boundedness is dropped, the conclusion of (3) may not be true. *O. M. Nikodým.*

Denjoy, Arnaud. Les dérivées. C. R. Acad. Sci. Paris **232**, 2053–2056 (1951).

Let $F(x)$ be a differentiable function in $[a, b]$ and set the derivative $F'(x)=f(x)$. The author shows the existence of a monotone increasing sequence of closed sets E_n in $[a, b]$ and of continuous functions $\varphi_n(x)$ such that $\varphi_n(x)$ is constant in each interval contiguous to E_n, $f(x)=\varphi_n(x)$ and $F(x)=\Phi_n(x)$ on E_n (where $\Phi_n(x)$ is a primitive of $\varphi_n(x)$), and $\varphi_n(x)\to f(x)$ in $[a, b]$. In this way the Darboux intermediate-value property of $f(x)$ is made evident. *A. Rosenthal* (Lafayette, Ind.).

Scorza Toso, Annamaria. Sulla derivazione di una funzione composta. Rend. Sem. Mat. Univ. Padova **21**, 198–201 (1952).

Let the function $z(x, y)$, defined in the rectangle R $(a\leqq x\leqq b, c\leqq y\leqq d)$, be continuous with respect to x and to y separately; let the functions $x=x(t)$, $y=y(t)$ be absolutely continuous in the interval I $(\alpha\leqq t\leqq\beta)$ such that almost everywhere in I $x'^2(t)+y'^2(t)>0$ holds; let the point $(x(t), y(t))$ be an interior point of R for every $t\in I$, and let $z(x, y)$ have first partial derivatives at this point for almost every t in I. It is proved that then the composite function $Z(t)=z(x(t), y(t))$ has an approximate derivative almost everywhere in I and that this derivative equals

$$z_x'(x(t), y(t))x'(t)+z_y'(x(t), y(t))y'(t)$$

almost everywhere. If $Z(t)$ is differentiable almost every-

where, then the usual formula of differentiation holds almost everywhere. *A. Rosenthal* (Lafayette, Ind.).

Kitagawa, Tosio. Random integrations. Bull. Math. Statist. 4, 15–21 (1950).

The author introduces the idea of "random integration", which might better be called "random approximate integration". The randomness is introduced in the method of approximation or in the choice of the function to be integrated, or both.

In the first type of random integration (or random approximation of integrals) a function $f(t)$ continuous on $0\leqq t\leqq 1$ is assumed to be fixed and one wishes to estimate $\int_0^1 f(t)dt$. The random variables are of the form

$$S_A(f)=\sum_{i=1}^{n}A_i f(t_i) \quad \text{where} \quad 0\leqq t_1\leqq\cdots\leqq t_n\leqq 1,$$

$$\sum_{i=1}^{n}A_i=1, \qquad A_i\geqq 0 \quad (i=1,\cdots,n).$$

The randomness is introduced in selecting the division-points $\{t_i\}$, $i=1, 2, \cdots, n$. Three different (random) ways to select the division points are mentioned and two different ways of choosing the A_i after the t_i have been selected are discussed. This gives six possible ways to form the sum $S_A(f)$. These six estimates are unbiased estimates of the integral and hence a comparison of the variances of the estimates is used as a criterion for the best estimate. Of course the variance will depend on the function $f(t)$.

In the second type of random integration (or approximate integration of random functions), a function space is given with a probability measure. The method of approximation by finite sums is fixed so that the sum $S_A(x)$ can be considered as a random variable as x varies in the space. $S_A(x)$ is used as an estimate of the integral $\int_0^1 x(t)dt$, which is now also a random variable, so that we have one functional approximating another. The variance of the error $\int_0^1 x(t)dt-S_A(x)$ is studied. In particular, the author uses the space of continuous functions $x(t)$ defined on $[0, 1]$ and such that $x(0)=0$, with Wiener measure.

The third type of random integration considered is a direct combination of the first two types, the function and the method of approximating its integral both being random.

R. H. Cameron and J. M. Shapiro.

Pták, Vlastimil. Proof of a theorem of Ward. Časopis Pěst. Mat. 76, 217–224 (1951). (Czech)

Given an additive real function $F(J)$ of the m-dimensional Euclidean interval J with the measure $|J|$, let $f(x, \alpha, \epsilon)$ denote the infimum of the ratio $F(J)/|J|$ for intervals J of parameter of regularity $\geqq\alpha$ and diameter $<\epsilon$ which contain the point x. The quantity $\inf_\alpha \sup_\epsilon f(x, \alpha, \epsilon)$ $(0<\alpha\leqq 1, \epsilon>0)$ is termed the lower derivative of $F(J)$ at the point x. The upper derivative is defined similarly. It was proved by A. J. Ward [Fund. Math. 28, 265–279 (1936)] that, in the set in which they are not infinite of opposite sign, these two upper and lower derivatives are almost everywhere finite and equal. The author derives this from a corresponding result for network derivatives, with the help of a simple geometrical lemma, in a straightforward manner.

L. C. Young (Madison, Wis.).

Kudryavcev, L. D., and Kaščenko, Yu. D. On the reduction of a multiple Lebesgue integral to an iterated one. Uspehi Matem. Nauk (N.S.) 7, no. 6(52), 211–212 (1952). (Russian)

Cafiero, Federico. Sull'inversione dell'ordine d'integrazione. Rend. Sem. Mat. Univ. Padova 21, 58–63 (1952).

The author proves the following theorem: Let $f(x, y)$ be measurable in the rectangle $R=I_1\times I_2$ (where I_1, I_2 are intervals on the x-axis and y-axis, respectively) and let $f(x, y)$ be summable with respect to x in I_1 for almost all y and summable with respect to y in I_2 for almost all x. If one of the two integrals

$$\int_G\int_H f(x, y)dydx, \qquad \int_H\int_G f(x, y)dxdy$$

exists for every pair of measurable sets $G\subset I_1$ and $H\subset I_2$, then the other one exists also and both are equal.

A. Rosenthal (Lafayette, Ind.).

Theory of Functions of Complex Variables

★Lavrent'ev, M. A., i Šabat, B. V. Metody teorii funkcii kompleksnogo peremennogo. [Methods of the theory of functions of a complex variable.] Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1951. 606+ii pp. 17.70 rubles.

This books stands somewhere between an introductory text on complex variables and a treatise. In the first two chapters the foundations are covered rapidly, but adequately for the remaining chapters. The chapter titles are: (I) Fundamental concepts; (II) Conformal mapping; (III) Boundary problems in the theory of functions and their applications; (IV) Variational principles in conformal mapping; (V) Applications of the theory of functions to analysis; (VI) The operational method and its applications; (VII) Special functions.

The book contains no exercises for the reader, and the references given are almost entirely to other textbooks. The authors have designed their work for the physicist or engineer with some previous knowledge of complex variables who wishes a quick review of the essentials leading to the more advanced topics in the later chapters. For this purpose the text is excellent; numerous specific examples are worked in complete detail, a wide selection of material is covered, the proofs are clear and concise, great attention is given to motivation, and the text includes over two hundred drawings. Among these are three-dimensional drawings showing a portion of the surface $u=|f(z)|$ for the special functions $\sin z$, $\tan z$, $\Gamma(z)$, $J_0(z)$, $\operatorname{sn} z$ $(k=.8)$, $F(z, k)$ (the inverse of $\operatorname{sn} z$), and the Hankel function $H_0^{(1)}(z)$. On each of these surfaces a set of level curves $|f(z)|=c_1$ and $\arg f(z)=c_2$ are shown. Unfortunately the text is marred by an unusually large number of misprints.

In chapter II a large number of mapping functions are developed in complete detail. Frequently the image regions are variable depending on a real parameter, and suitable approximations for the mapping functions are developed for small values of the parameter. The authors obtain an approximate means of varying the Schwarz-Christoffel transformation, in order to produce a rounding of one of the corners.

In chapter III known results on integrals of Cauchy type are used to solve (a) a Hilbert-Privalov boundary value problem, and (b) a mixed boundary value problem in which $f(z)$ is analytic in the upper half plane, and $\Re f(z)$ and $\Im f(z)$ are prescribed on alternate intervals of the real axis. These

results are in turn used to solve a Riemann-Hilbert boundary value problem and a special case of the Tricomi problem.

The chapter on special functions contains an elegant unified treatment of orthogonal polynomials based on the fact that for all the standard sets of orthogonal polynomials, the weight function $\rho(x)$ satisfies the Pearson differential equation $\rho'/\rho=(\alpha_0+\alpha_1x)(\beta_0+\beta_1x+\beta_2x^2)$.

The authors make a number of interesting statements, a few of which are reproduced here as a matter of record. The Cauchy-Riemann equations are called the D'Alembert-Euler equations. Liouville's theorem is called the Liouville-Cauchy theorem. The Weierstrass theorem on essential singularities was first proved by Yu. V. Sohockiĭ (1868) who also obtained the results on integrals of Cauchy type (1873) usually associated with Plemelj and Privalov. Paletina's formula was first obtained by D. A. Grave (1896). The Nyquist method of locating the roots of a polynomial [Bell System Tech. J. 11, 126–147 (1932)] was first suggested by Vyšnegradskiĭ (1877). The Schwarz inequality is called the Bunyakovskiĭ inequality. The Laguerre polynomials of zero order and the Hermite polynomials were first studied by Čebyšev (1859). *A. W. Goodman.*

H **✶Green, S. L. The theory and use of the complex variable. An introduction.** 2d ed. Sir Isaac Pitman & Sons, Ltd., London, 1950. viii+136 pp. 15 shillings.

For a review of the first edition see these Rev. 11, 91.

✶Knopp, Konrad. Elements of the theory of functions. Translated by Frederick Bagemihl. Dover Publications, Inc., New York, 1953. 140 pp. Paperbound $1.25. Clothbound $2.25.

on order

Translated from Elemente der Funktionentheorie [Sammlung Göschen Bd. 1109, de Gruyter, Berlin, 1936.

Kreisel, G. Some elementary inequalities. Nederl. Akad. Wetensch. Proc. Ser. A. 55 = Indagationes Math. 14, 334–338 (1952).

Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$, where a_n can be effectively calculated to any desired accuracy, and let $|a_n|<Mr^{-n}$. For these functions of a complex variable the author gives constructive versions of fundamental theorems on uniqueness of the expansion and on the existence of zeros. Let $|f(0)|>c_1>0$. Then there exists a p such that for any distinct $z_0, z_1, \cdots, z_p$, where $|z_i|<3R/8$, there exists a constructive positive lower bound for max $|f(z_i)|$. Also, for any $\eta>0$, the zeros of such an $f(z)$ in $|z|<(3/8-\eta_1)R$, for some $\eta_1\leqq\eta$, may be approximated by a sequence of nested regions, with diameters convergent to zero. *D. Nelson* (Washington, D. C.).

Talbot, A. The roots of certain determinantal equations. Math. Gaz. 36, 270–272 (1952).

I. If a rational function of x has poles and zeros (including a possible pole or zero at infinity) which are all real, simple, and strictly alternate [cf. Moore, Math. Gaz. 34, 94–98 (1950); these Rev. 12, 472], then its partial fraction expansion $k_\infty x+c+\sum_{r=1}^{n}k_r/(\gamma_r-x)$ has coefficients k_r (including k_∞ if non-zero) all of one sign, while its reciprocal has expansion coefficients all of the opposite sign. Conversely, any such sum having coefficients all of the same sign is equal to a rational function of the above kind. II. If an odd rational function $f(\lambda)$ of a complex variable $\lambda=\sigma+i\omega$ has poles and zeros which are simple and strictly alternate on the imaginary axis, then the partial-fraction expansions of the form $c\lambda+\sum_{r=1}^{p}k_r\lambda/(\omega_r^2+\lambda^2)$ with $0\leqq\omega_1^2<\omega_2^2<\cdots<\omega_p^2$ for the function and its reciprocal, have coefficients k_r, and c if non-zero, all of the same sign. Conversely, any such expansion represents an odd rational function of the above kind. *E. Frank* (Chicago, Ill.).

Carleson, Lennart. On the zeros of functions with bounded Dirichlet integrals. Math. Z. 56, 289–295 (1952).

Let $z_n=r_ne^{i\theta_n}$, $0<r_n<1$. We say $f(z)=1+a_1z+\cdots$ belongs to C if $f(z_n)=0$, $n=1, 2, \cdots$, and $\sum_{m=1}^{\infty}m|a_m^2|<\infty$. Lokki [Ann. Acad. Sci. Fennicae Ser. A. I. Math.-Phys. no. 39 (1947); these Rev. 9, 277] claimed to prove that C is empty if and only if $\sum_{n=1}^{\infty}(1-r_n)$ diverges. The writer shows that if $\theta_n=0$, $n=1, 2, \cdots$, this condition is necessary. He goes on to prove that if the r_n are assigned and θ_n are left arbitrary, a sufficient condition for C to be always empty is that (*) $\sum\log(1-r_n)^{-k}<\infty$ for some $k>-1$ and a necessary condition is that (*) hold for every $k<-1$. In view of a variety of unproved statements the reviewer was unable to follow the proofs of these latter results. *W. K. Hayman* (Exeter).

Agmon, Shmuel. On the singularities of Taylor series with reciprocal coefficients. Pacific J. Math. 2, 431–453 (1952).

Gegeben sei das Funktionselement (1) $f(z)=\sum_{n=0}^{\infty}a_nz^n$ mit $a_n\neq0$ $(n=0, 1, \cdots)$ und $\lim_{n\to\infty}|a_n|^{1/n}=1$, und es sei $f_{-1}(z)=\sum_{n=0}^{\infty}a_n^{-1}z^n$ gesetzt. Soula [Bull. Soc. Math. France 56, 36–49 (1928)] hatte gezeigt, dass im Falle reeller a_n ein enger Zusammenhang zwischen den Singularitäten von $f(z)$ und $f_{-1}(z)$ auf $|z|=1$ besteht. Verf. bemerkt zunächst, dass dabei die Voraussetzung reeller a_n überflüssig ist und stellt sich dann die Aufgabe, auch die ausserhalb von $|z|=1$ liegenden Singularitäten von $f(z)$ und $f_{-1}(z)$ in Beziehung zu setzen. Die Ecken des zu $f(z)$ gehörigen Hauptsterns seien $\rho(\theta)e^{i\theta}$ $(0\leqq\theta<2\pi)$, und mit $\rho(\theta)$ [bzw. $\rho_{-1}(\theta)$] werde die "Sternfunktion" von $f(z)$ [bzw. $f_{-1}(z)$] bezeichnet. Unter der Annahme, dass (1) in der längs $1\leqq x<\infty$ geschlitzten Ebene regulär ist $(z=x+iy)$, lässt sich der zu $f_{-1}(z)$ gehörige Hauptstern vollständig angeben: Es existieren Zahlen α, β mit $0\leqq\alpha$, $\beta\leqq\pi/2$ (deren Berechnungsmöglichkeit angedeutet wird), sodass

$$\rho_{-1}(\theta)=\min\{\exp[(\tan\alpha)\theta], \exp[(\tan\beta)(2\pi-\theta)]\} \quad (0<\theta<2\pi)$$

und $\rho_{-1}(0)=1$ ist. Das Ergebnis kann verwendet werden, um eine Aussage über die Lage der Singularitäten von $f_{-1}(z)$ auch in dem Falle zu machen, in dem von $f(z)$ nur bekannt ist, dass es in dem von $|z|=\rho$ und $1\leqq x\leqq\rho$ $(\rho>1)$ berandeten endlichen Gebiet der z-Ebene regulär ist. *D. Gaier.*

Pidduck, F. B. Some integral representations of an analytic function. Quart. J. Math., Oxford Ser. (2) 3, 222–226 (1952).

Let p and q be two points in the complex plane, $\varphi(u)$ a function defined along a curve joining p and q, $\Phi(z)$ an entire function such that

$$\int_p^q\varphi(u)\Phi(uz)du=\frac{1}{1-z}, \quad |z|<1.$$

Let $f(z)$ be single-valued and analytic in the ring $b<|z|<a$ and C a curve in this ring surrounding $|z|=b$. We have then for $b<|z|<a$,

$$2\pi if(z)=\int_p^q\varphi(u)du\int_C\frac{f(t)[\Phi(uz/t)+\Phi(ut/z)]dt}{t}-\int_C\frac{f(t)dt}{t}.$$

Various interesting special cases are considered.
G. Szegö (Stanford, Calif.).

Vekua, N. P. On a problem of the theory of functions of a complex variable. Doklady Akad. Nauk SSSR (N.S.) 86, 457–460 (1952). (Russian)

Let L be a set of simple, closed, suitably smooth, disjoint curves in the plane, limiting a connected bounded domain D^+; D^- is the complement of $D^+ + L$. The author solves the problem (an extension of a problem of Hilbert) of finding a piecewise analytic vector $\Phi(z)$, of finite order at ∞, so that on L one has $\Phi^+ = A\Phi^- + B\bar{\Phi}^- + g$, where the vector g and square matrices A, B are assigned of class H (Hölder). The method used is that of integral equations in the sense of principal values. *W. J. Trjitzinsky* (Urbana, Ill.).

Ahiezer, N. I. On entire transcendental functions of finite degree having a majorant on a sequence of real points. Izvestiya Akad. Nauk SSSR. Ser. Mat. 16, 353–364 (1952). (Russian)

The author proves a generalization of Cartwright's theorem [Quart. J. Math., Oxford Ser. (1) 7, 46–55 (1936)] that an entire function $f(z)$ of exponential type less than π, bounded on the points $\{n\}_{-\infty}^{\infty}$, is bounded on the real axis. He compares $f(z)$ with a function $\omega_0(z)$ which is of genus 0, satisfies $|\omega_0(x)| \geqq 1$ on the real axis, and has zeros $\alpha_k + i\beta_k$ such that $\sum 1/|\beta_k| < \infty$. To such an $\omega_0(z)$ and any positive p correspond real numbers x_k and an entire function $\theta(z)$ of exponential type p such that if $f(z)$ is of type $\sigma < p$ and $|f(x_k)| \leqq |\omega_0(x_k)|$, then $|f(x)| \leqq C|\theta(x)|$, where C depends only on ω_0, σ and p; all the zeros of $\theta(z)$ lie in the upper half plane. The points x_k and function $\theta(z)$ are, in fact, given fairly explicitly in terms of ω_0. We can replace $\omega_0(z)$ by $\omega(z)$, of the same kind and having the same absolute value for real z, but with all its zeros in the upper half plane. Then the x_k are the solutions of

$$\arg \omega(x_k) + px_k + \lambda = \tfrac{1}{2}\pi + k\pi,$$

where λ is a fixed real number; defining

$$\Omega(z) = \tfrac{1}{2}\{\omega(z)e^{i(pz+\lambda)} + \bar{\omega}(z)e^{-i(pz-\lambda)}\},$$

we have for $\theta(z)$ an entire function of exponential type p such that $|\theta(x)|^2 = \sum_{-\infty}^{\infty} \{\Omega(z)/(z - x_k)\}^2$. The proof depends on representing $f(z)$ by an interpolation formula based on the points x_k (which are the real zeros of $\Omega(z)$).

A more general result, that $|f(x_k)| \leqq |\omega_0(x_k)|\alpha(x_k)$ implies $|f(x)| \leqq D|\theta(x)|\alpha(x)$ for the class of continuous functions $\alpha(x)$ for which $\alpha(x) \geqq 1$, $\alpha(x_1 + x_2) \leqq \alpha(x_1)\alpha(x_2)$, and $\int_{-\infty}^{\infty}(1+x^2)^{-1} \log \alpha(x)dx < \infty$, depends on an auxiliary result of V. A. Marčenko (at present unavailable outside the USSR) [Zapiski Mat. Otd. Fiz.-Mat. Fak. i Harkov. Mat. Obšč. (4) 22, 115–125 (1950)]. *R. P. Boas, Jr.*

Ahiezer, N. I. On a family of entire functions of finite degree and a problem of Čebyšev. Izvestiya Akad. Nauk SSSR. Ser. Mat. 16, 459–468 (1952). (Russian)

The author defines a special entire function $G(z; a, k)$, of exponential type p, based on the mapping of a rectangle into a half plane (hence expressed in terms of elliptic functions) and shows that it has the following extremal property. For all entire functions $\psi(z)$ of exponential type p (at most), with $\psi(x_1) = \xi_1$, $\psi(x_2) = \xi_2$, $-1 < x_1 < x_2 < 1$, if $MG(z; a, k)$ takes these same values at x_1, x_2, then $\sup|\psi(x)|$ (over the real axis with $(-1, 1)$ deleted) exceeds M unless $\psi(z) \equiv MG(z; a, k)$. *R. P. Boas, Jr.* (Evanston, Ill.).

Ahiezer, N. I. On entire functions of finite degree deviating least from zero. Mat. Sbornik N.S. 31(73), 415–438 (1952). (Russian)

The author considers the following general extremal problem for entire functions. Let $P(z)$ and $Q(z)$ be polynomials of degrees n and $n-1$; minimize the deviation of $f(z) = Q(z) + P(z)g(z)$ (i.e., $\sup|f(x)|$, $-\infty < x < \infty$) for all $g(z)$ of exponential type not exceeding p. He supposes that P and Q are real and $P(z) \geqq 0$ on the real axis. Special cases (previously treated otherwise by S. Bernstein) are to find the extremal $f(z)$ for $f(0) = a_0$, $f'(0) = a_1$ or for $f(c) = \gamma$, $f(\bar{c}) = \bar{\gamma}$; here $P(z) = z^2$ and $Q(z)$ is linear. A more difficult problem is to find the extremal $f(z)$ for $f(0) = a_0$, $f'(0) = f'''(0) = 0$, $f''(0) = a_2$ [S. Bernstein, Izvestiya Akad. Nauk SSSR. Ser. Mat. 13, 111–124 (1949); these Rev. 11, 22]. The author gives a more complete treatment than Bernstein's. He also discusses the still more difficult problem with data $f(0) = f''(0) = 0$, $f'(0) = a_1$, $f'''(0) = a_3$; the solution involves the function $G(z; a, k)$ of the paper reviewed above.

The main theorems involve the following definitions. A real sequence $\{\lambda_k\}$ belongs to $T(p, m)$ if it is the set of zeros of an entire function $\Omega(z)$ of exponential type such that $|\Omega(z)| > c|z|^{-m}e^{p|y|}$ for $|\pm\pi/2 - \arg z| < \delta$, $|z| > 1$, and $|\Omega'(\lambda_j)| > D(|\lambda_j| + 1)^{-m}$; it belongs to $T^*(p, m)$ if in addition $\liminf |\Omega(iy)|\,|y|^m e^{-p|y|}$ is finite and not zero. A Chebyshev set for a function $f(x)$ is a maximal simply ordered set $\{x_k\}$ such that $f(x)$ takes values $\pm L$ (L is its deviation) alternately on the x_k. Let K be the class of functions $Q(z) + P(z)g(z)$ (as above), and K_R the subclass of functions which are real on the real axis. Then if $P(z)$ is of degree $2l+2$, if $F(z)$ belongs to K_R and takes values $\pm L$ on the points of a sequence of $T(p, 2l)$, and if $|Q(z)| \neq L$ at the real roots of $P(z)$, then $F(z)$ is the unique extremal function in K. In the opposite direction, if the Chebyshev set of $F(z)$ belongs to $T^*(p, 2h)$, $h \geqq l+1$, and if $|Q(z)| \neq L$ at the real roots of $P(z)$, then $F(z)$ is not extremal.

The entire functions of exponential type whose Chebyshev sets belong to $T(p, 0)$ are of the form $L \sin(pz + \alpha)$. Those whose Chebyshev sets belong to $T(p, 2)$ are the much more complicated functions $G(z; a, k)$ mentioned above. *R. P. Boas, Jr.* (Evanston, Ill.).

Bernštein, S. N. On antimajorants. Izvestiya Akad. Nauk SSSR. Ser. Mat. 16, 497–502 (1952). (Russian)

The author has previously [Doklady Akad. Nauk SSSR (N.S.) 65, 117–120 (1949); these Rev. 11, 23] introduced the concept of an antimajorant, which he now defines as a nonnegative function $H(x)$ such that for arbitrary a, b, N there is an entire function $G_p(z)$ of positive exponential type p such that $|G_p(x)| \leqq H(x)$ while $|G_p'(x)| > N$ somewhere on (a, b). He stated a necessary and sufficient condition for a class of functions $H(x)$ to be antimajorants, and proved the necessity; here he proves the sufficiency. The class of functions admitted consists of $|H(x)|$, where $H(z)$ is an entire function of exponential type, such that $|H(x)|$ is "almost increasing" (i.e., $|H(x)| \leqq k|H(\lambda x)|$ for some $k \geqq 1$ whenever $\lambda \geqq 1$). Such a function is shown to be an antimajorant if the roots $\alpha_n + i\beta_n$ of $H(z)$ satisfy $\sum|\beta_k|/(\alpha_k^2 + \beta_k^2) = \infty$. In fact, $|H(x)|$ is a "weak weight function," which means that every continuous function vanishing at $\pm\infty$ can be approximated with weight $|H(x)|$ by entire functions of arbitrarily small exponential type; a weak weight function is easily seen to be an antimajorant. Corollaries: The product of an antimajorant and the absolute value of an entire function of exponential type is an antimajorant; if $H(z)$ is an entire function of exponential type, $H(x) > 0$, and $\int^{\infty} x^{-2} \log |H(x)H(-x)|dx = \infty$, then $|\int_0^x p(x)H(x)dx|$ is an antimajorant if $p(x) > c > 0$.

R. P. Boas, Jr. (Evanston, Ill.).

Kubo, Tadao. Bounded analytic functions in a doubly connected domain. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **27**, 41–45 (1952).

The author considers for the case $n=2$ the following extremal problem treated by Ahlfors [Duke Math. J. **14**, 1–11 (1947); these Rev. **9**, 24]: Let D denote a region of connectivity n containing $z=\infty$ and let B denote the class of analytic functions f in D which are of modulus less than one in D and vanish at ∞. With $f=\sum_1^\infty a_k z^{-k}$, it is required to maximize $|a_1|$ for $f \in B$. The author determines the extremal value of $|a_1|$ with the aid of elliptic functions.

M. H. Heins (Paris).

R.-Salinas, Baltasar. Note on the region of values of a schlicht function. Revista Mat. Hisp.-Amer. (4) **12**, 223–228 (1952). (Spanish)

Using variational methods developed by Schaeffer, Spencer, and Grad [Coefficient regions for schlicht functions, Amer. Math. Soc. Coll. Publ., vol. 35, New York, 1950; these Rev. **12**, 326], the author determines the region of values for $f(z)$ and $zf'(z)/f(z)$, for the family of functions $f(z)=z+\cdots$ schlicht in $|z|<1$. For such functions he obtains the inequalities

$$|\log f(z)(1-|z|^2)/z| \leqq \log \frac{1+|z|}{1-|z|},$$

and

$$|\log zf'(z)/f(z)| \leqq \log \frac{1+|z|}{1-|z|}.$$

There are a number of disturbing misprints.

A. W. Goodman (Lexington, Ky.).

Robertson, M. S. A coefficient problem for functions regular in an annulus. Canadian J. Math. **4**, 407–423 (1952).

The following theorems are proved. I. Let $f(z)=\sum_{-\infty}^{\infty} a_n z^n$ be regular for $0 \leqq \rho \leqq |z| \leqq 1$, and on each circle $|z|=r$, $\rho<r<1$, let $\Im f(z)$ change sign $2p$ times where p does not depend on r. Then, for $n>p$,

$$|a_n-\bar{a}_{-n}| \leqq \sum_{k=0}^{n} \Delta(p, k, n)\,|a_k-\bar{a}_{-k}|,$$

where

$$\Delta(p, 0, n)=\prod_{\nu=1}^{p}(n^2-\nu^2)/(p!)^2,$$

$$\Delta(p, k, n)=2\prod_{\nu=0,\,\nu\neq k}^{p}(n^2-\nu^2)/(p+k)!(p-k)! \quad \text{for } k>0.$$

II. Let $f(z)=\sum_0^\infty a_n z^n$ be regular and multivalent of order p for $|z|<1$, and let $\partial \Re f(re^{i\theta})/\partial\theta$ change sign $2p$ times on each circle $|z|=r$ where $\rho<r<1$. Then, for $n>p$,

$$|a_n| \leqq \sum_{k=1}^{p} \frac{2k(n+p)!}{(p+k)!(p-k)!(n-p-1)!(n^2-k^2)}|a_k|.$$

All these estimates are sharp. II follows from I and generalises a previous result of the author ($p=1$) [Amer. J. Math. **58**, 465–472 (1936)].

W. W. Rogosinski.

Wigner, E. P. Simplified derivation of the properties of elementary transcendentals. Amer. Math. Monthly **59**, 669–683 (1952).

A study is made of some of the properties of a special class of meromorphic functions $R(z)$ for which the product of the imaginary part of $R(z)$ and the imaginary part of z is non-negative in the complex z-plane [see also E. P. Wigner, Ann. of Math. (2) **53**, 36–67 (1951); these Rev. **12**, 490]. These properties include, in particular, two expansion theorems for an arbitrary R function. Using the more commonly known general properties of analytic functions which may be established without use of the properties of the elementary transcendental functions, the author gives a derivation of the properties of the trigonometric and exponential functions from a unified point of view, and without appeal to the properties of these functions for real arguments. This is done by defining the tangent as the simplest periodic R function. The general expansions as infinite sum or product for an R function give in this case the well-known expansions for the elementary transcendental functions.

M. S. Robertson.

Hervé, Michel. A propos d'un mémoire récent de M. Noshiro: Nouvelles applications de sa méthode. C. R. Acad. Sci. Paris **232**, 2170–2172 (1951).

Let $f(z)$ be meromorphic in a region D of the complex plane with boundary C; let E be a subset of C of logarithmic capacity zero; and let ζ_0 be a point of E. Let $\Delta(\zeta)$ denote the cluster set of $f(z)$ at $\zeta \in C$ and $\Gamma_E(\zeta_0)$ denote the set $\bigcap_{r>0} M_r$, where M_r is the closure of the union $\bigcup_\zeta \Delta(\zeta)$, where ζ ranges over the intersection of the set $C-E$ with the circle $|\zeta-\zeta_0|<r$. It was shown by Noshiro [J. Math. Soc. Japan **1**, 275–281 (1950); these Rev. **13**, 224] that if D is simply connected and if the open set $\Delta(\zeta_0)-\Gamma_E(\zeta_0)$ is not empty, then $w=f(z)$ assumes every value belonging to any component of this set, with at most two exceptions, infinitely often in any neighborhood of ζ_0. The author extends this result to certain cases in which D is no longer necessarily simply connected. For a different recent extension, see Yosida [Proc. Japan Acad. **27**, 268–274 (1951); these Rev. **14**, 365].

W. Seidel (Princeton, N. J.).

Collingwood, Edward F. Conditions suffisantes pour l'inversion de la seconde inégalité fondamentale de la théorie des fonctions méromorphes. C. R. Acad. Sci. Paris **235**, 1182–1184 (1952).

Dans cette note, l'auteur donne des conditions suffisantes pour l'inversion de la seconde inégalité de la théorie de Nevanlinna. Sa méthode généralise des travaux de Teichmüller [Deutsche Math. **2**, 96–107 (1937)] et de H. L. Selberg [Comment. Math. Helv. **18**, 309–326 (1946); ces Rev. **8**, 23]; elle utilise les notations de Nevanlinna et les définitions données par l'auteur dans quatre notes précédentes [mêmes C. R. **227**, 615–617, 709–711, 749–751, 813-815 (1948); ces Rev. **10**, 244, 363] développées dans les Trans. Amer. Math. Soc. **66**, 308–346 (1949) [ces Rev. **11**, 94], en particulier les notions de valence $p_\nu(r, a, \sigma(a, r))$, de niveau $\sigma(r)$, de niveau fermé et de niveau d'indice zéro par rapport à $f(z)$. Adaptant ces définitions antérieures et désignant par $P(r, a, \sigma(a, r))$ le maximum de $p_\nu(r, a, \sigma(a, r))$ pour les ν qui correspondent à des domaines qui coupent la circonférence $|z|=r$, il définit l'ensemble $E(a, \sigma(a, r), p(r))$ où $p(r)$ est une fonction n'ayant que des discontinuités ordinaires et arrive au théorème suivant. Soit $f(z)$ méromorphe pour $|z|<R\leqq\infty$, $T(r, f)$ non bornée. Supposons qu'il existe une fonction $\sigma(r)$ d'indice zéro, une fonction $p(r)=o(T(r, f)/\log r)$ si $R=\infty$ ou $p(r)=o(T(r, f))$ si $R\leqq\infty$ et un nombre $B<R$, tels que, étant donné un système fini de nombres a_i, $1\leqq i\leqq q$, on ait pour tout a appartenant à l'ensemble complémentaire des a_i, sup $CE(a, \sigma(a, r), p(r))\leqq B$. On a alors

$$2T(r, f) \leqq N_1(r)+\sum_{i=1}^{q} m(r, a_i)+\tau(r)$$

avec $\tau(r)=o(T(r,f))$ si $R=\infty$, et

$$\tau(r)=\log[1/(R-r)]+o(T(r,f))$$

si $R<\infty$. On a posé

$$N_1(r)=N\left(r,\frac{1}{f'}\right)+2N(r,f)-N(r,f').$$

Ce théorème se réduit à celui de Selberg si $\sigma(r)$ et $p(r)$ sont des constantes. *G. Valiron* (Paris).

Collingwood, Edward F. Relation entre la distribution des valeurs multiples d'une fonction méromorphe et la ramification de sa surface de Riemann. C. R. Acad. Sci. Paris **235**, 1267–1270 (1952).

L'auteur utilise les résultats de sa note précédente dans l'étude de la ramification de la surface de Riemann S décrite par les valeurs de la fonction méromorphe $f(z)$ représentées sur la sphère. Les notations étant toujours les mêmes, les propriétés de $E(a,\sigma(r),p(r))$ et de $V(a,\sigma(r),p(r))$ caractérisent la ramification de S au voisinage du point a. Les fonctions $\sigma(r)$ et $p(r)$ sont assujetties à des conditions: $\log(1/\sigma(r))$ est supposée infiniment petite par rapport à $T(r,f)$ lorsque r tend vers R et $p(r)$ infiniment petite par rapport à $T(r,f)/\log r$ si $R=\infty$ ou par rapport à $T(r,f)$ si $R<\infty$. On sait que la condition $\sup CE(a,\sigma(r),p(r))<R$ entraîne que la déficience supérieure $\Delta(a)$ (de Valiron) est nulle. L'auteur suppose que, si $R<\infty$,

$$\lim_{r\to R}\frac{T(r,f)}{-\log(R-r)}=\infty,$$

et qu'il existe un $B<R$ tel que $\sup CE(a,\sigma(r),p(r))\leqq B$ pour tout a. Alors, en outre de $\Delta(a)=0$, on a

$$\liminf_{r\to R}\frac{N_1(r)}{T(r,f)}=2.$$

De ce théorème, l'auteur déduit que, si la limite précédente est inférieure à 2 et si $p(r)$ satisfait à la condition indiquée, il existe une fonction $\Sigma(r)$ non croissante et tendant vers zéro lorsque r tend vers R et une valeur a telles que

$$\sup CE(a,\Sigma(r),p(r))=R.$$

Ceci donne une indication sur la mesure de l'ordre de ramification infini de la surface S au point a. *G. Valiron.*

Umezawa, Toshio. Analytic functions convex in one direction. J. Math. Soc. Japan **4**, 194–202 (1952).

Let $f(z)=z+\sum_2^\infty a_nz^n$ be meromorphic for $|z|<1$. If there exists a real number $\alpha\geqq 3/2$ for which

$$(1)\qquad \alpha>1+\Re\frac{zf''(z)}{f'(z)}>\frac{-\alpha}{2\alpha-3},\quad |z|<1,$$

then $f(z)$ is regular and univalent in $|z|<1$, and $f(z)$ maps $|z|=r$, for every $r<1$, into a contour which is convex in one direction. In proving this theorem, the author shows that (1) implies that

$$(2)\qquad \int_0^{2\pi}\left|1+\Re\frac{zf''(z)}{f'(z)}\right|d\theta<4\pi,\quad z=re^{i\theta},\quad r<1,$$

and that (2) implies (3) that $f(z)$ is convex in one direction. The reviewer remarks that the latter part of the theorem, that (2) implies (3), was shown earlier by A. Rényi [see Publ. Math. Debrecen **1**, 18–23 (1949); these Rev. **11**, 92]. The author also shows that if $f(z)$, $f(0)=0$, $f'(0)=1$, is regular and univalent in $|z|<1$, then $f(z)$ is convex in one direction for $|z|<4-(13)^{\frac{1}{2}}$. The author extends his results to p-valent functions. *M. S. Robertson.*

Ozaki, Shigeo, Ono, Isao, and Ozawa, Mitsuru. On the pseudo-meromorphic mappings on Riemann surfaces. Sci. Rep. Tokyo Bunrika Daigaku. Sect. A. **4**, 211–213 (1952).

The authors extend their prior work on pseudo-meromorphic functions [same Rep. **4**, 203–205 (1951); these Rev. **13**, 835] to the case of pseudo-meromorphic functions on a Riemann surface. *M. H. Heins* (Paris).

Ozaki, Shigeo, and Ono, Isao. Second principal theorem of pseudo-meromorphic functions. Sci. Rep. Tokyo Bunrika Daigaku. Sect. A. **4**, 214–221 (1952).

The authors establish analogues of the second fundamental theorem of Nevanlinna for two- and three-dimensional pseudo-meromorphic maps. *M. H. Heins.*

Ozaki, Shigeo, Kashiwagi, Sadao, and Tsuboi, Teruo. On the function-theoretic identities of continuous mappings. Sci. Rep. Tokyo Bunrika Daigaku. Sect. A. **4**, 238–242 (1952).

Let $w(z)=u(x,y)+iv(x,y)$ be a single-valued function of $z=x+iy$ in a domain D_z except for a finite number of poles (a point z_0 is a pole if $\lim_{z\to z_0}w(z)=\infty$ for all paths leading to z_0) and let u_x, u_y, v_x, v_y exist and be continuous in D_z except at the poles. Let z_k, $k=1,2,\cdots,m$, be the roots of $w(z)-w_0=0$ in D_z and let

$$J(z)=\begin{vmatrix}u_x & u_y\\ v_x & v_y\end{vmatrix},\quad n(D_z,w_0)=m_1-m_2,$$

where m_1 is the number of roots z_k for which $J(z_k)\geqq 0$, m_2 is the number of roots z_k for which $J(z_k)<0$. Then

$$(1)\qquad \iint_{|w|\leqq\infty}n(D_z,w_0)\,du\,dv=\iint_{D_z}J(z)\,dx\,dy.$$

It is shown that $n(D_z,w_0)$ is invariant as w varies on a continuous curve which does not touch the image Γ_w of the boundary Γ_z of D_z by $w=w(z)$. If w crosses Γ_w, the saltus of $n(D_z,w_0)$ is an integer. With the use of this property of $n(D_z,w_0)$ and (1) the authors state that a series of theorems obtained for pseudo-meromorphic functions by S. Ozaki, I. Ono, and M. Ozawa [same Rep. **4**, 157–160, 161–168 (1951); these Rev. **13**, 453] may be proved and extended for continuous mappings. *M. S. Robertson.*

Ozaki, Shigeo, Kashiwagi, Sadao, and Tsuboi, Teruo. On the function-theoretic identities on the continuous mapping in the three dimensional space. Sci. Rep. Tokyo Bunrika Daigaku. Sect. A. **4**, 243–249 (1952).

The concepts used in the paper reviewed above are extended to three dimensions with resulting extensions of their theorems to 3-space. *M. S. Robertson.*

Komatu, Yûsaku, and Ozawa, Mitsuru. Conformal mapping of multiply connected domains. II. Kōdai Math. Sem. Rep. **1952**, 39–44 (1952).

[For part I see same Rep. **1951**, 81–95; these Rev. **13**, 734.] The authors establish the existence of conformal mappings of doubly-connected regions onto canonical regions consisting of the extended plane less two slits. The proofs are based upon the construction of analytic functions in an annulus which have properties appropriate for the present purpose. *M. H. Heins* (Paris).

Unkelbach, Helmut. Die konforme Abbildung echter Polygone. Math. Ann. **125**, 82–118 (1952).

The author extends the concept of a polygon by admitting any simply-connected domain which contains no branch-points in its interior and which is bounded by a finite number of straight lines, half-lines, rectilinear segments, or isolated points. He then derives a formula, generalizing the classical Schwarz-Christoffel's, for a function mapping the upper half-plane onto a thus extended polygonal domain which is supposed to cover the point at infinity finitely many times. Let an n-gon with the vertices E_ν ($\nu=1, \cdots, n$) of respective apertures $\alpha_\nu\pi$ and with p poles be given. Then the function $z=f(\zeta)$ mapping the upper half of the ζ-plane onto it in such a manner that $E_1, \cdots, E_{n-1}$, and E_n correspond to $\kappa_1, \cdots, \kappa_{n-1}$, and ∞, respectively, is of the form

$$f(\zeta)=C\int_{\zeta_0}^{\zeta}\prod_{\nu=1}^{n-1}(\tau-\kappa_\nu)^{\alpha_\nu-1}e^{R(\tau)}\prod_{\nu=1}^{p}(\tau-\lambda_\nu)^{-2}(\tau-\bar{\lambda}_\nu)^{-2}d\tau,$$

where $R(\tau)$ is a determinate rational function with real coefficients which possesses no poles except at κ_ν and ∞, and the λ_ν ($\nu=1, \cdots, p$) denote the image of the poles contained in the polygon. Conversely, a function of the above form yields the mapping of $\Im\zeta>0$ onto a polygon when and only when the system of p equations

$$\sum_{\nu=1}^{n-1}\frac{\alpha_\nu-1}{\lambda_\mu-\kappa_\nu}+R'(\lambda_\mu)-\sum_{\nu=1}^{p}\frac{2}{\lambda_\mu-\bar{\lambda}_\nu}-\sum_{\substack{\nu=1\\ \nu\neq\mu}}^{p}\frac{2}{\lambda_\mu-\lambda_\nu}=0 \quad (\mu=1, \cdots, p)$$

are satisfied.

In certain special cases, the problem of determining the parameters contained in the main formula is treated in some detail; in connection with this, relations are derived between the distributions of asymptotic values and of poles of a locally schlicht meromorphic function. The author states an application of the formula to a weighted mean of the curvature of the boundary curve for the mapping of the exterior of a circular disc onto an exterior domain of an analytic contour. *Y. Komatu* (Tokyo).

Renggli, Heinz. Un théorème de représentation conforme. C. R. Acad. Sci. Paris **235**, 1593–1595 (1952).

Estimates, based on extremal length, of the conformal representation of a canonical domain. *P. R. Garabedian.*

✱Bieberbach, L. Conformal mapping. Translated by F. Steinhardt. Chelsea Publishing Co., New York, 1953. vi+234 pp. $2.25.

Translation of Einführung in die konforme Abbildung, 4th ed. [Sammlung Göschen, Bd. 768, de Gruyter, Berlin, 1949].

Ozawa, Mitsuru. Classification of Riemann surfaces. Kōdai Math. Sem. Rep. **1952**, 63–76 (1952).

The author extends the classification problem of Riemann surfaces by considering functions u which satisfy the elliptic partial differential equation $\Delta u=Pu$, where P is positive except at a countable number of zeros not accumulating at any point of the surface. The Green's function, harmonic measure, kernel function, bounded functions and functions with a finite Dirichlet integral are defined with respect to this equation. The existence of non-constant functions belonging to these classes is studied, both on the whole surface and in a boundary neighborhood. Inclusion relations known to be valid for harmonic functions continue to hold in the present case. *L. Sario* (Cambridge, Mass.).

Tsuji, Masatsugu. Maximal continuation of a Riemann surface. Kōdai Math. Sem. Rep. **1952**, 55–56 (1952).

Proof of Bochner's theorem concerning the existence of a maximal continuation of a given Riemann surface along the lines of that given by the reviewer [Ann. of Math. (2) **43**, 280–297 (1942); these Rev. **4**, 77]. *M. H. Heins.*

Röhrl, Helmut. Die Elementartheoreme der Funktionenklassen auf geschlossenen Riemannschen Flächen. Math. Nachr. **7**, 65–84 (1952).

In a previous paper [Math. Nachr. **6**, 355–384 (1952); these Rev. **13**, 736] the author began the investigation of the function class k described by functions $r(w, z)\exp V$, where r is an element of the function field $C(w, z)$ on a closed Riemann surface $\mathfrak{F}$ and $V=\int\varphi(w, z)dz$, $\varphi\in C(wz)$, $C=$complex field. In the present paper he defines elementary functions and differentials, and "effective" principal parts in terms of which the partial fraction decompositions of functions and differentials are obtained. The theory is extended to functions of two variables, and various expansion and interchange theorems are proved. A reduction theorem provides a finite linear basis for functions on k, whose singularities are confined to a certain finite set of places. The reduction theorem is proved again, this time over an arbitrary base field to which has been adjoined a certain finite set of numbers which depend on $\mathfrak{F}$ and on $\varphi(w, z)$. *J. Lehner* (Los Alamos, N. M.).

Fourès, Léonce. Le problème des translations isothermes ou construction d'une fonction analytique admettant dans un domain donné une fonction d'automorphie donnée. Ann. Inst. Fourier Grenoble **3** (1951), 265–275 (1952).

Let Δ_1 and Δ_2 be disjoint simply connected regions of a Riemann surface R, which are in bi-uniform correspondence under a function ψ satisfying certain conditions concerning the mapping of the boundaries. Suppose there exists a region Γ containing both Δ_1 and Δ_2, and a mapping Φ of Γ on a circle such that the transform of ψ by Φ (i.e., $\Phi\psi\Phi^{-1}$ in the sense of composition) is in the class of ψ. The author proves that there is then a bi-uniform mapping g of Γ on a plane region G of the w-plane such that $g\psi g^{-1}$ carries $w\in G$ into $w+k$ ($k=$constant). He then considers the situation in which there are infinitely many simply connected regions Δ_j bounded by Jordan arcs, and bi-uniform mappings ψ_{ij} of Δ_j on Δ_i. Under these circumstances there exists a region Γ on the surface R, Γ contains every Δ_j, and a function f regular in Γ with the property that $f(\zeta)=f(\zeta')$ if and only if ζ and ζ' are conformal images in some regions Δ_p, Δ_q. The methods employed are geometric and do not involve the use of harmonic functions. [Cf. earlier papers of the author, e.g., Ann. Sci. Ecole Norm. Sup. (3) **68**, 1–64 (1951); C. R. Acad. Sci. Paris **232**, 1894–1895 (1951); these Rev. **12**, 691; **13**, 125.] *J. Lehner* (Los Alamos, N. M.).

Garabedian, P. R., and Spencer, D. C. Complex boundary value problems. Trans. Amer. Math. Soc. **73**, 223–242 (1952).

Soit M un domaine de l'espace numérique complexe $(z_1, \cdots, z_m)$ ayant une frontière régulière ∂M. Soit $\hat{d}$ (resp. $\hat{\delta}$) la composante de type $(1, 0)$ (resp. de type $(-1, 0)$) de l'opérateur de différentiation d (resp. de codifférentiation δ pour la métrique euclidienne); enfin soient ζ la forme différentielle extérieure $dz_1\wedge\cdots\wedge dz_m$ et $T\Phi$ la forme différentielle induite sur ∂M par la forme Φ définie au voisinage de ∂M. On résout le problème aux limites suivant: Trouver une forme différentielle A de type $(p, 0)$ (forme pure)

satisfaisant au système (1): (a) $T(\zeta A)=T\theta$, où θ est une forme donnée de type (p, m) définie au voisinage de ∂M; (b) $\bar{\delta}\bar{\partial}A=0$ dans M; (c) $A=\bar{\delta}Q$ dans M où Q est une forme de type $(p+1, 0)$. L'équation de Poisson fournit une solution de (a) et (b); (c) conduit à une équation intégrale singulière. Une combinaison de la théorie de Fredholm et d'une méthode de projection orthogonale donne la solution unique du système (1). L'existence des formes pures de Green G_p et de Neumann N_p en résulte ainsi que leur relation avec la forme noyau de Bergman K_p; d'où la solution des problèmes aux limites en termes de G_p et N_p. On obtient également une formule de Cauchy généralisée pour les formes pures $\bar{d}$ et $\bar{\delta}$-fermées. L'application la plus importante de cette théorie est le développement, à l'aide des équations intégrales, du formalisme des problèmes aux limites pour les équations de Cauchy-Riemann à m variables complexes [P. R. Garabedian, J. Analyse Math. 1, 59–80 (1951); ces Rev. 13, 25]; en particulier $K_m=\bar{\partial}G_{m-1}$ [P. R. Garabedian, Ann. of Math. (2) 55, 19–33 (1952); ces Rev. 13, 736]. La technique utilisée pour résoudre (1) généralise celle de la théorie du potentiel; elle est formellement très voisine de celle employée pars G. F. D. Duff et D. C. Spencer [ibid. (2) 56, 128–156 (1952); ces Rev. 13, 987]. *P. Dolbeault* (Paris).

Lelong, Pierre. La convexité et les fonctions analytiques de plusieurs variables complexes. J. Math. Pures Appl. (9) 31, 191–219 (1952).

Eine reellwertige Funktion $V(X)$—der Wert $-\infty$ ist zugelassen—in einem Gebiet D des Raumes von n komplexen Veränderlichen, $X_k=x_k+ix_k'$, heisst plurisubharmonisch in D (pseudokonvex bei K. Oka), wenn 1) V in jedem kompakten Teil von D nach oben beschränkt ist; 2) jeder Schnitt mit einer analytischen Ebene eine subharmonische Funktion [siehe T. Radó, Subharmonic functions, Springer, Berlin, 1937] liefert. Verf. gibt zusätzlich äquivalente Definitionen mit Hilfe des Mittelwertsatzes auf Polyzylindern sowie mittels der Hermiteschen Form $\sum(\partial^2V/\partial X_i\partial\bar{X}_j)A_i\bar{A}_j$ an. Durch diese Hermitesche Form wird jeder plurisubharmonischen Funktion in ihrem ganzen Existenzgebiet eine Kählersche Metrik zugeordnet. Zu den plurisubharmonischen Funktionen gehört insbesondere die Klasse $l_n(D)$ der Funktionen $C\log|f(X)|$, wo $f(X)$ als Funktion der n komplexen Veränderlichen regulär in D ist. Verf. untersucht die Frage, ob $\overline{l_n(D)}=L_n(D)$ ist, wo $L_n(D)$ die Gesamtheit der in D plurisubharmonischen Funktionen bedeutet. Für $n=1$ trifft es zu, für $n\geqq 2$ hängt sie mit den Eigenschaften des Randes von Regularitätsgebieten zusammen. Siehe den Beweis von Oka für das Levische Problem im Falle $n=2$ [Tôhoku Math. J. 49, 15–52 (1942); diese Rev. 7, 290].

Im 2. Teil beschäftigt sich Verf. mit dem Fall, dass die betrachteten plurisubharmonischen Funktionen zurückführbar sind auf Funktionen, die in gewissen euklidischen Räumen sich konvex verhalten. Das ist insbesondere so, wenn man Funktionen betrachtet, die in Tuben regulär sind; Tuben sind Gebiete, die alle Transformationen

$$X_k'=X_k+it_k,\quad -\infty<t_k<+\infty,\quad k=1,\cdots,n,$$

in sich zulassen. [Siehe auch S. Bochner and W. T. Martin, Several complex variables, Princeton, 1948; diese Rev. 10, 366; K. Stein, Math. Ann. 114, 543–569 (1937); Hans Bremermann, Schr. Math. Inst. Univ. Münster 5 (1951).] *H. Behnke* (Münster).

Lelong, Pierre. Equivalence de certaines propriétés de pseudo-convexité. C. R. Acad. Sci. Paris 235, 594–596 (1952).

Eine reelle, nach oben halbstetige Funktion im Raume von n komplexen Veränderlichen R^n nennt Verf. plurisubharmonisch (bei K. Oka pseudokonvex), wenn sie sich auf jeder analytischen Ebene subharmonisch verhält. Verf. führt dann (analog zur Regulärkonvexität) den Begriff des in bezug auf plurisubharmonische Funktionen konvexen (P-konvexen) Gebietes ein. Es werden notwendige und hinreichende Bedingungen für plurisubharmonische Gebiete D aufgestellt, z.B.: Für $M\in D$ ist $-\log\delta(M)$ plurisubharmonisch, wenn $\delta(M)$ die euklidische Entfernung des Punktes M vom Rande von D bedeutet. Ein Gebiet D heisst lokal P-konvex, wenn jeder Punkt M von R^n eine P-konvexe Umgebung U hat, sodass $D\cap U$ auch P-konvex ist. Dann gilt: Ein schlichtes, lokal P-konvexes Gebiet ist P-konvex (Analogon zum grossen Satz von E. E. Levi und Oka). *H. Behnke* (Münster).

Fréchet, Maurice. Sur deux familles de fonctions analogues à la famille des fonctions analytiques. C. R. Acad. Sci. Paris 235, 1585–1587 (1952).

The author presents some analogies between the family of analytic functions $X+iY$ of the variable $x+iy$, a family D of functions $X+jY$ of the variable $x+jy$, where $1\times j=j\times 1=j$, $j\times j=0$, and a family P of functions $X+kY$ of the variable $x+ky$, where $1\times k=k\times 1=0$, $k\times k=k$. In place of the Cauchy-Riemann equations one has for the family D that $\partial X/\partial x-\partial Y/\partial y=\partial X/\partial y=0$ and for the family P that $\partial X/\partial y=\partial Y/\partial x=0$. *C. J. Titus.*

Theory of Series

Boyd, A. V., and Hyslop, J. M. A definition for strong Rieszian summability and its relationship to strong Cesàro summability. Proc. Glasgow Math. Assoc. 1, 94–99 (1952).

Let k, p, and p' be constants such that $k>0$, $p\geqq 1$, $p^{-1}+p'^{-1}=1$ and $kp'>1$. A series $\sum a_n$ is said to be strongly evaluable to s by the Riesz method (R, k) of order k if the transform $C_k(x)=\sum_{n<x}(1-n/x)^k a_n$ converges to s as $x\to\infty$ and, in addition,

$$\lim_{x\to\infty} x^{-1}\int_1^x |uC_k'(u)|^p du=0.$$

It is shown that $\sum a_n$ is strongly evaluable (R, k) if and only if it is strongly evaluable by the Cesàro method (C, k). *R. P. Agnew* (Ithaca, N. Y.).

Agnew, Ralph Palmer. Arithmetic means and the Tauberian constant .474541. Acta Math. 87, 347–359 (1952).

Let $s_n=\sum_{k=0}^n u_k$, $M_n=(s_0+\cdots+s_n)/(n+1)$. It is well-known that (1) $\lim\sup_{n\to\infty}|M_n-s_{p_n}|\leqq B\lim\sup|nu_n|$ if $p_n=n$ and $B=1$. In this and a previous paper [Proc. London Math. Soc. (3) 2, 369–384 (1952), these Rev. 14, 160] the author solves the following problem: What is the least value of B for which (1) is true for each sequence u_n and some sequences of indices p_n (which may depend on the u_n)? It is shown here that (1) holds if $B=B_0$ is the unique solution of $\exp(-\frac{1}{2}\pi B_0)=B_0$ and the p_n are chosen properly with $B_0^2n\leqq p_n\leqq n$; in the paper cited above it was proved that (1) is false for properly chosen u_n and each sequence

p_n if $B<B_0$. The present proof is indirect and based on an estimate of the sum $\sum_{k=0}^{n} s_k$ under the assumption that $|M_n - s_k| \geqq B_0+\epsilon$ for all $\lambda n \leqq k \leqq n$, $0<\lambda<1$. There are other similar results of which we quote that the sign "$\leqq$" in (1) in the theorem stated may be replaced by "$<$" if the sequence s_n is bounded. The author remarks that for other methods of summation (in particular, for Abel's method) the minimal value of the corresponding constant is not known. *G. G. Lorentz* (Toronto, Ont).

Agnew, Ralph Palmer. Approximation by use of kernels originating from Abel transforms of series. Comment. Math. Helv. 26, 171–179 (1952).

The author discusses the closedness of approximation of functions $z(t)$, $-\infty<t<+\infty$, of the class Lip 1 by singular integrals connected with the Abel transform. In this way he obtains a simple proof of his result [Duke Math. J. **12**, 27–36 (1945); Ann. of Math. (2) **50**, 110–117 (1949); these Rev. **7**, 12; **10**, 291] that $A_0=0.9680\cdots$ is the smallest constant A with

$$\limsup_{n\to\infty} \left|\sum_{k=0}^{\infty} r_n{}^k u_k - \sum_{k=0}^{n} u_k\right| \leqq A \limsup |nu_n|,$$

where the $r_n \to 1$ may be chosen arbitrarily after the u_n are given. *G. G. Lorentz* (Toronto, Ont.).

Vernotte, Pierre. La sommation des séries asymptotiques de seconde espèce. C. R. Acad. Sci. Paris **235**, 1469–1471 (1952).

Carlson, Fritz. Contributions à la théorie des séries de Dirichlet. IV. Ark. Mat. **2**, 293–298 (1952).

[For parts I–III see Ark. Mat. Astr. Fys. **16**, no. 18 (1922); **19A**, no. 25 (1926); **23A**, no. 19 (1933).] Suppose that the Dirichlet series $\sum_1^\infty a_n \exp(-\lambda_n s)$ have a half-plane of convergence and set

$$F(x, t) = \sum_{\lambda_k<x} (x-\lambda_k) a_k \exp(-\lambda_k it),$$

so that $f(\sigma+it)=\sigma^2\int_0^\infty e^{-\sigma x}F(x, t)\,dx$, $\sigma>\max(0, \sigma_0)$. In order that the function $f(s)$ defined by the series be holomorphic and $\leqq 1$ in modulus for $\sigma>0$, it is necessary that $|F(x, t)| \leqq x$ for each $x>0$, t real, and sufficient that this inequality hold for $x=\lambda_{n+1}$, $n=1, 2, 3, \cdots$. The coefficients a_k of such a series are subject to several inequalities such as: $\sum_1^\infty |a_k|^2 \leqq 1$, $|a_k| \leqq 1-|a_1|^2$, $n=2, 3, \cdots$, and if $2\lambda_p<\lambda_1+\lambda_n$, then $|a_n| \leqq 1-|a_1|^2-|a_2|^2-\cdots-|a_p|^2$. In particular, if $\lambda_n=\log n$, one has $\sum_1^\infty |a_n| n^{-\rho} \leqq 1$, where ρ is the root of the equation $\zeta(\rho)=1+(1+|a_1|)^{-1}$. Finally, let

$$F(x_1, x_2, \cdots)=a_1+\sum c_j x_j+\sum c_{jk} x_j x_k+\cdots$$

be a function such that each section $F_m(x_1, x_2, \cdots, x_m)$ obtained by putting $x_{m+1}=x_{m+2}=\cdots=0$ has the property of being regular and at most one in modulus in the domain $|x_1|<1$, $|x_2|<1$, $\cdots$, $|x_m|<1$. Then the sum of the moduli of the terms of the series does not exceed one in the domain $|x_k|<\rho_k$, $k=1,2,3,\cdots$, where $\prod_1^\infty (1-\rho_k)^{-1}<1+(1+|a_1|)^{-1}$. *E. Hille* (Nancy).

Farinha, João. On two theorems of Pincherle. Revista Fac. Ci. Univ. Coimbra **21**, 161–165 (1952). (Portuguese)

I. Let $f_1, f_2, \cdots$, be functions of a complex variable, and

$$\phi=\phi_1^\infty \frac{1}{f_n} \sim \frac{1}{f_1+} \frac{1}{f_2+} \frac{1}{f_3+}\cdots.$$

The the continued fraction converges uniformly if $|f_1| \geqq 1+\alpha$, $\alpha>0$, and $|f_n| \geqq 2$, $n>1$. Here $|\phi| \leqq 1/\alpha$, and $1/(|f_1|+1) \leqq |\phi| \leqq 1/(|f_1|-1)$. II. The zeros of the denominator of each approximant of the continued fraction

$$\phi_1^\infty \frac{1}{a_n z+b_n},$$

where a_n and b_n are complex numbers, $a_n \neq 0$, satisfy the condition

$$\min \frac{|b_k|-2}{|a_k|} < |z| < \max \frac{|b_k|+2}{|a_k|}, \quad k=1, 2, \cdots, n.$$

(Reviewer's note: The convergence in Theorem I follows immediately from the criteria of Pringsheim.)
E. Frank (Chicago, Ill.).

Fourier Series and Generalizations, Integral Transforms

Young, Frederick H. Transformations of Fourier coefficients. Proc. Amer. Math. Soc. **3**, 783–791 (1952).

Let a_j, b_j be the Fourier coefficients of a function $f \in L_p$, and let $T=(a_{kj})$ be an infinite matrix, the first row and column of which consist (for convenience) of zeros. $T \in (L_p)$ means that, whenever $f \in L_p$, then the series $A_k=\sum_0^\infty a_{kj}a_j$ and $B_k=\sum_0^\infty a_{kj}b_j$ converge and represent Fourier coefficients of a function $F \in L_p$. If $1<p<\infty$, a complete characterisation of such matrices $T \in (L_p)$ is given. Also $F=T(f)$ is then a linear bounded transformation on L_p, and $T^* \in (L_{p'})$, where $1/p+1/p'=1$ and T^* is the transpose of T. In the two cases $p=1$ and $p=\infty$ similar results are obtained under the additional assumption on T that $\sum_{j=0}^\infty |a_{kj}|<M$ uniformly for all k. For the special case of T being the matrix of the arithmetical means compare G. H. Hardy [Messenger of Math. **58**, 50–52 (1928)] and R. Bellman [Bull. Amer. Math. Soc. **50**, 741–744 (1944); these Rev. **6**, 125].
W. W. Rogosinski (Newcastle-upon-Tyne).

Berkovitz, Leonard D. Double Sturm-Liouville expansions. Duke Math. J. **19**, 567–574 (1952).

In connection with the classical work by Haar [Math. Ann. **69**, 331–371 (1910)] it is shown that the double cosine series and any corresponding double Sturm-Liouville expansion of a function which is integrable in a square are uniformly restrictedly equiconvergent in closed subsets of the square. This means that the difference $s_{mn}-S_{mn}$ of the partial sums tends uniformly to zero when m and n are restricted by a condition $a<mn^{-1}<a^{-1}$ $(a>0)$. An analogous result for summability $(C, 1, 1)$ and f in L^p $(p>1)$, is also proved. [Cf. Mitchell, Amer. J. Math. **65**, 616–636 (1943); these Rev. **5**, 96.] *L. Gårding* (Lund).

Stewart, C. A. Fourier expansions. Proc. Glasgow Math. Assoc. **1**, 76–93 (1952).

Beginning with linear differential equations with constant coefficients, the author employs step-functions, Laplace transforms, contour integrals, and residue theory to establish the representation of a function of bounded variation on the interval $(0, 2\pi)$ by its Fourier series.
R. V. Churchill (Ann Arbor, Mich.).

Kalinovskaya, S. S. On the convergence of the deviations from the polynomials of mean power approximations to best approximations. Doklady Akad. Nauk SSSR (N.S.) 84, 437–440 (1952). (Russian)

Let $\varphi(x)=c_0v_0(x)+c_1v_1(x)+\cdots+c_pv_p(x)$ where the c_k are arbitrary reals and the $v_k(x)$ are bounded, uniformly continuous functions on a Lebesgue measurable set E of finite positive measure in R_n. Suppose that the v_k are metrically linearly independent, i.e., $mE>mE_\varphi$ for each φ where $E_\varphi=\{x|x\varepsilon E, \varphi(x)=0, \sum c_k^2\neq 0\}$, and that $\psi(h)>0$ when $h>0$, where $\psi(h)=\inf_{x\varepsilon E, h_0\geq r\geq h} m(E\cdot S(x,r))/mS(x,r)$ with $S(x,r)$ denoting a sphere of radius r and center x. Let φ_m and φ_0 be φ's minimizing respectively the expressions $\delta_m(\varphi)=\{(mE)^{-1}\int_E|\varphi(x)|^m dx\}^{1/m}$ and $\delta_0(\varphi)=\sup_{x\varepsilon E}|\varphi(x)|$ under the restriction that $c_0=1$. It is shown that under these conditions $\delta_0(\varphi_m)\rightarrow\delta_0(\varphi_0)$ as $m\rightarrow\infty$, the rapidity of convergence being determined by the behavior of $\psi(h)$ and the moduli of continuity of the v_k. This is a generalization of the corresponding result due to Remez [same Doklady (N.S.) 60, 199–202 (1948); these Rev. 10, 529] on the line where the restriction on $\psi(h)$ is unnecessary.

G. Klein (South Hadley, Mass.).

Hunt, G. A. Random Fourier transforms. Trans. Amer. Math. Soc. 71, 38–69 (1951).

This paper is devoted to the study of series of the type

$$Z(t)=\sum X_n e^{i\lambda_n t},$$

where the X_n's are independent and subject to conditions $E\{X_n\}=0$, $\sum E\{X_n^2\}<\infty$. Also analogous "random integrals" of the form $\int e^{i\lambda t}X(d\lambda)$ are studied. Although the methods used are closely related to those of Paley and Zygmund [Proc. Cambridge Philos. Soc. 26, 337–357, 458–474 (1930); 28, 190–205 (1932)] they are used in a powerful way to yield a wealth of new results. A typical result is that if

$$\sum E\{X_n^2\}|\lambda_n|^{2\alpha}<\infty$$

for some $\alpha\leqq 1$ then almost certainly $Z(t)=o(\log t)$ as $t\rightarrow\infty$ and $Z(t)$ has (with probability 1) the modulus of continuity of the form const.$\times h^\alpha(\log h^{-1})^{1/2}$. Of particular interest are extensions of such results to "random integrals". Theorem 12, for example, contains as a special case the law of the iterated logarithm for the Wiener process. The ingenious use of ergodic theory is particularly noteworthy.

M. Kac (Ithaca, N. Y.).

Reiter, H. J. Investigations in harmonic analysis. Trans. Amer. Math. Soc. 73, 401–427 (1952).

This paper is concerned with various questions about the L^1 group algebra of a locally compact abelian group G and the related Fourier transform theory. It is divided into three parts which are mutually independent. Part 0 (unnumbered in the paper) gives a proof of the known theorem that an analytic function of the Fourier transform of a function in $L^1(G)$ is again such a Fourier transform. The proof is modeled after Carleman's proof for the case in which G is the real line and is less algebraic in spirit than the standard proofs for general groups. The two principal results of part I are as follows. If I is a closed ideal in $L^1(G)$ and I_0 is a regular maximal ideal in I, then $I_0=I\cap J$ where J is a regular maximal ideal in $L^1(G)$; moreover, the canonical mapping of I/I_0 onto the complex numbers is norm-preserving. If G' is a homomorphic image of G, then $L^1(G')$ is homomorphic to $L^1(G)$ by a homomorphism whose kernel is the set of all functions whose Fourier transforms vanish on the dual of G'; moreover, the isomorphism of $L^1(G')$ with the quotient of $L^1(G)$ by this kernel is norm-preserving. In part II a recent generalization of Wiener's general Tauberian theorem due to Mandelbrojt and Agmon [Acta Sci. Math. Szeged 12, Pars B, 167–176 (1950); these Rev. 11, 660] is generalized from the real line to arbitrary locally compact Abelian groups. A substantially equivalent generalization has recently been given by Helson [Ark. Mat. 1, 497–502 (1952); these Rev. 14, 246]. The present author's proof, however, was found without knowledge of Helson's work and is of a rather different nature. As in part 0 the methods used are more akin to those used in classical analysis than to those customary in functional analysis.

G. W. Mackey (Cambridge, Mass.).

Stone, M. H. On the foundations of harmonic analysis. Comm. Sém. Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.] Tome Supplémentaire, 207–227 (1952).

This paper appeared earlier in Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] 21, no. 17 (1952); these Rev. 14, 17.

Zitarosa, Antonio. Sulla formula di inversione per la trasformata di Hankel. Rend. Accad. Sci. Fis. Mat. Napoli (4) 18 (1951), 268–272 (1952).

Hankel's inversion theorem is usually proved for functions which are integrable in $(0,\infty)$ (and satisfy some local conditions): the infinite integrals involved are absolutely convergent. In the present paper, the theorem is proved for functions which are integrable in $(0,b)$ for all $b>0$, of bounded variation in (a,∞) for all $a>0$, and tend to zero as $x\rightarrow\infty$. The infinite integrals involved are in general simply convergent Riemann integrals or "improper" Lebesgue integrals.

A. Erdélyi (Pasadena, Calif.).

Bose, S. K. On Laplace transform. Math. Z. 56, 84–93 (1952).

The author obtains a number of integral formulae connecting functions and their Laplace transforms. The results are then applied to derive various integral transforms connecting special functions.

S. Agmon (Jerusalem).

Polynomials, Polynomial Approximations

Mohr, Ernst. Berichtigung zu meiner Arbeit: Der sogenannte Fundamentalsatz der Algebra als Satz der reellen Analysis. Math. Nachr. 6, 385–386 (1952).

Discussion of a previous paper [Math. Nachr. 6, 65–69 (1951); these Rev. 13, 938].

A. C. Schaeffer.

Sz.-Nagy, Gyula. Über Polynome, deren Nullstellen auf einem Kreis liegen. Acta Math. Acad. Sci. Hungar. 2, 157–164 (1951). (Russian summary)

Given a polynomial $f(z)=(z-a_1)(z-a_2)\cdots(z-a_n)$, a point ζ, and an arbitrary set $p_1, p_2, \cdots, p_n$ of positive numbers. By a polar polynomial $g(z)$ of $f(z)$ with respect to the pole ζ, the author means

$$g(z)=f(z)\sum_{k=1}^{n} p_k(\zeta-a_k)/(z-a_k).$$

When $p_1=p_2=\cdots=p_n=1$, $g(z)$ reduces to the polar polynomial $P=nf+(\zeta-z)f'$ as usually defined. The author proves that, if the a_k and ζ lie on a circle K, then all the zeros of $g(z)$ lie on K and together with ζ separate the a_k.

This generalizes a theorem due to Laguerre concerning the zeros of P. Furthermore, he proves that, if the zeros of $f(z)$, all lying on a circle K, are separated by the zeros of the polynomial $h(z)=(z-b_1)(z-b_2)\cdots(z-b_n)$, all likewise lying on K, then each of the polynomials $[h(z)/(z-b_k)]$ for $k=1, 2, \cdots, n$ is proportional to a polar polynomial with pole at $\zeta=b_k$. Both theorems are first proved for K the real axis and then generalized by applying a linear transformation. *M. Marden* (Milwaukee, Wis.).

Tietze, Heinrich. Über eine Klasse von Polynomen, die diejenigen mit lauter positiven Nullstellen umfasst. Math. Nachr. 8, 7–12 (1952).

A real polynomial $f(x)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}a_kx^{n-k}$ with $a_0=1$ and all other $a_k>0$ is said to be "medial-monotonic" if the "elementary symmetric mean-value" $b_k=a_k^{1/k}$ satisfies the inequality $b_k\geqq b_{k+1}$ for $k=1, 2, \cdots, n-1$. It is well-known that polynomials all of whose zeros are positive are "medial-monotonic". The author furnishes some other examples of medial-monotonic polynomials. By use of the theorem on the arithmetic and geometrical means, he shows that, if $c>0$ and $f(x)$ is medial-monotonic, then so is $F(x)=f(x)(x-c)$. *M. Marden* (Milwaukee, Wis.).

Bandić, Ivan. Geometrische Deutung eines Satzes. Bull. Soc. Math. Phys. Macédoine 2, 121–124 (1951). (Serbo-Croatian. German summary)

Es wird bewiesen dass der Satz, "Wenn zwei Polynome

$$F(x)=C_0x^{n-1}+C_1x^{n-2}+\cdots+C_{n-2}x+C_{n-1},$$
$$f(x)=x^n+b_1x^{n-1}+\cdots+b_{n-1}x+b_n$$

gegeben sind, wobei $f(x)$ nur einfache Nullstellen a_i hat, dann gilt $\sum_{i=1}^{n}F(\alpha_i)/f'(\alpha_i)=C_0$", eine Folge eines bekannten Satzes von Newton über Asymptoten algebraischer Kurven ist. *Author's summary.*

Gatteschi, Luigi. Limitazione dell'errore nella formula di Hilb e una nuova formula per la valutazione asintotica degli zeri dei polinomi di Legendre. Boll. Un. Mat. Ital. (3) 7, 272–281 (1952).

In several papers the author has made more precise various estimates involving unspecified constants by determining numerical values for these constants. In the present paper this is done for Hilb's estimates

$$\sigma=\left(\frac{\sin\theta}{\theta}\right)^{1/2}P_n(\cos\theta)-J_0[(n+\tfrac12)\theta]=\begin{cases}\theta^{1/2}O(n^{-3/2})\\ \theta^2O(1)\end{cases},$$

holding for $n^{-1}\leqq\theta\leqq\pi/2$, $0<\theta\leqq n^{-1}$, respectively. The following inequalities are proved:

$$|\sigma|<\begin{cases}0.358\theta^{-1/2}n^{-3/2}+0.394\theta^{1/2}n^{-3/2}\\ 0.09\theta^2\end{cases},$$

holding for $\pi/2n\leqq\theta\leqq\pi/2$ and $0<\theta\leqq\pi/2n$, respectively. As an application the following inequality for the zeros $\theta_{n,r}$ of $P_n(\cos\theta)$ (ordered in increasing order) is obtained:

$$0<\frac{j_{0,r}}{n+\frac12}-\theta_{n,r}<\frac{16+37r}{16n^4},\quad r\leqq[n/2].$$

Here $j_{0,r}$ is the rth zero of $J_0(x)$. *G. Szegö.*

Orts, J. M.[a]. On the integral formula for the Legendre polynomials. Revista Mat. Hisp.-Amer. (4) 12, 201–205 (1952). (Spanish)

The integral formula for Legendre polynomials is discussed [cf. also the author's preceding article, Collectanea Math. 3, 105–120 (1950); these Rev. 13, 129]. *E. Frank* (Chicago, Ill.).

Tricomi, Francesco G. A class of non-orthogonal polynomials related to those of Laguerre. J. Analyse Math. 1, 209–231 (1951). (English. Hebrew summary)

The author has introduced earlier the polynomials $l_n(x)=(-1)^nL_n^{(x-n)}(x)$ of degree $[n/2]$ where $L_n^{(\alpha)}$ is the usual symbol for the Laguerre polynomials. They appear, among other instances, in a certain asymptotic expansion of the incomplete gamma-function studied by the author [Ann. Mat. Pura Appl. (4) 31, 263–279 (1950); these Rev. 13, 939]. In the present paper a systematic study of the $l_n(x)$ is undertaken, including their generating function, various representations, asymptotic behavior for large n, etc. It is proved that all zeros of $l_n(x)$ are real, simple, and, with the exception of the first (which is 0), greater than 1. The generalization $u=(-1)^nL_n^{(x-y-n)}(x)$ satisfies the parabolic differential equation

$$(\eta+\xi)u_{\xi\xi}+(\eta-\xi)u_\xi+nu=0$$

where $x=\xi+\eta$, $y=\xi-\eta+1-n$. *G. Szegö.*

Tricomi, Francesco G. La seconda soluzione dell'equazione di Laguerre. Boll. Un. Mat. Ital. (3) 7, 1–4 (1952).

The second solution of the differential equation of the Laguerre polynomials is expressed in terms of certain Laguerre polynomials and of the incomplete gamma-function. *G. Szegö* (Stanford, Calif.).

MacRobert, T. M. An expansion in terms of associated Legendre functions. Proc. Glasgow Math. Assoc. 1, 13–15 (1952).

The author obtains an explicit expression for the sum of the series

$$(\mu^2-1)^{\frac12 m}\sum_{n=0}^{\infty}h^nP_n^{-m}(\mu),$$

and corrects a statement regarding this series made in Hobson's The theory of spherical and ellipsoidal harmonics [Cambridge, 1931, p. 105]. *A. Erdélyi.*

Ikenberry, E., and Rutledge, W. A. Convergence of expansions in the Hermite polynomials $H_n(hw)$. J. Math. Physics 31, 180–183 (1952).

The domain of convergence of an expansion in terms of the Hermite polynomials $H_n(z)$ is a strip symmetrical to the real axis whose width can be determined by a formula similar to that of Cauchy-Hadamard [cf. G. Szegö, Orthogonal polynomials, Amer. Math. Soc. Colloq. Publ., v. 23, New York, 1939, p. 246; these Rev. 1, 14]. The author is expanding a given analytic function $f(z)$ in terms of $H_n(hz)$ where h is a complex constant. This constant can be fixed in various ways so as to adjust the domain of convergence (depending on h) to the special nature of the function $f(z)$. *G. Szegö* (Stanford, Calif.).

Geronimus, Ya. L. On the orthogonal polynomials of V. A. Steklov. Izvestiya Akad. Nauk SSSR. Ser. Mat. 16, 469–480 (1952). (Russian)

A Steklov polynomial system is an orthonormal set $\{\hat\varphi_k(x)\}$ $(k=0, 1, \cdots)$ on $[-1, 1]$ corresponding to a (non-negative) weight function $w(x)$, which is uniformly bounded on $[-1, 1]$ (or on some subinterval thereof). For such a system Steklov established that every function $F(x)$ satisfying a Lipschitz condition of order α $(0<\alpha\leqq1)$ has a Fourier expansion in the $\hat\varphi_k$-polynomials that is convergent to $F(x)$ everywhere in the interval of uniform boundedness. This property suggests the usefulness of finding sufficient

conditions in order that an orthonormal set be a Steklov set; and this is the purpose of the present work.

Let $\|f\|=\{(2\pi)^{-1}\int_0^{2\pi}|f(\theta)|^2d\theta\}^{\frac{1}{2}}$ be the usual norm for the (real- or complex-valued) function $f(\theta)\in L_2(0,2\pi)$. Suppose f is periodic, and let $\omega_2(f;\delta)$ be its integral modulus of continuity: $\omega_2(f;\delta)=\sup_{0<h\leq\delta}\|f(\theta+h)-f(\theta)\|$; then $f(\theta)$ is said to belong to class Lip $(\alpha,2)$ if $\omega_2(f;\delta)\leq M\delta^\alpha$ $(0<\alpha\leq1)$. The fundamental result is the theorem: $\{\hat{\varphi}_k(x)\}$ is a Steklov set on the whole interval $[-1,1]$ if $p(\theta)=w(\cos\theta)|\sin\theta|$ is bounded above and below: $0<m_1\leq p(\theta)\leq m_2$, and if $p(\theta)\in$ Lip $(\alpha,2)$ with $\frac{1}{2}\leq\alpha\leq1$.

The proof makes use of some lemmas, which we state: (I) If $G_n(z)$ is a polynomial of degree $\leq n$ such that (a) $|G_n(z)|\leq M_n$, $|G_n(e^{i\theta_0})|=M_n$ in $|z|\leq1$, then (b) $|G_n(re^{i\theta_0})|\geq\frac{1}{2}M_n$ for $r=1-(1/2n)$. (II) If $\varphi(z)$ is analytic in $|z|<1$ and $\varphi(z)\in H_2$, and if $G_n(z)$ satisfies (a), then (c) $M_n\leq2[|\varphi(re^{i\theta_0})|+2\delta_n\sqrt{n}]$ where $r=1-(1/2n)$ and $\delta_n=\|\varphi-G_n\|(r=1)$. (III) Let $\varphi(z)\in H_2$ and suppose its values on $|z|=1$ (as approached radially: $r\to1$) are bounded p.p. Suppose also that for each n there is a polynomial $G_n(z)$ that, in the metric of L_2, approximates to φ to order $O(n^{-\alpha})$, $\frac{1}{2}\leq\alpha\leq1$. Then the sequence $\{G_n(z)\}$ is uniformly bounded in $|z|\leq1$. (IV) Let $p(\theta)\in$ Lip $(\alpha,2)$ $(0<\alpha\leq1)$ and let $p(\theta)\geq m_1>0$ p.p. on $[0,2\pi]$. Then $\pi(e^{i\theta})\in$ Lip $(\alpha,2)$ where

$$\pi(z)=h^{1/2}\exp\left\{\frac{1}{4\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\log p(\theta)d\theta\right\},\ |z|<1,$$

and

$$h=\exp\left\{(2\pi)^{-1}\int_0^{2\pi}\log p(\theta)d\theta\right\}.$$

Moreover, $p(\theta)=h|\pi(e^{i\theta})|^{-2}$ p.p. This function $\pi(z)$ is used to obtain: (V) Let $\{P_n(z)\}$ be the polynomials with leading coefficient unity, orthogonal relative to the distribution

$$d\sigma(\theta):\quad \frac{1}{2\pi}\int_0^{2\pi}P_n(e^{i\theta})\cdot\overline{P_m(e^{i\theta})}d\sigma(\theta)=\begin{cases}0, & m\neq n;\\ h_n>0, & m=n.\end{cases}$$

Let $\{\hat{P}_n(z)\}$ be the corresponding orthonormal set:

$$\hat{P}_n(z)=h_n^{-1/2}P_n(z).$$

Suppose $\sigma(\theta)$ is absolutely continuous, and that $\sigma'(\theta)=p(\theta)\in$ Lip $(\alpha,2)$, $\frac{1}{2}\leq\alpha\leq1$, and $0<m_1\leq p(\theta)\leq m_2$ p.p. on $[0,2\pi]$. Then $\{\hat{P}_n(z)\}$ is uniformly bounded on $|z|\leq1$.

If $\{\varphi_k(x)\}$ is the orthonormal set on $[-1,1]$ corresponding to $w(x)$, and $\{\hat{P}_n(z)\}$ the set on $|z|=1$ corresponding to the distribution $d\sigma(\theta)$, with $\sigma'(\theta)=p(\theta)=w(\cos\theta)|\sin\theta|$, then the two sets are related by

$$\hat{\varphi}_n(x)=\frac{1}{[2\pi(1-a_{2n-1})]^{1/2}}\cdot\frac{\hat{P}_{2n}(z)+\hat{P}^*_{2n}(z)}{z^n}\quad(z=x+(x^2-1)^{1/2}),$$

where $P_{n+1}(z)=zP_n(z)-\bar{a}_nP_n^*(z)$ and $a_n\to0$. The fundamental theorem then follows from (V). The article closes with the statement of a number of consequences of the theorem. *I. M. Sheffer* (State College, Pa.).

Campbell, Robert. Sommes de Fejér et moyennes de Césaro pour les développements d'une fonction en série de polynomes orthogonaux usuels. C. R. Acad. Sci. Paris **235**, 773–776 (1952).

Starting out from the Christoffel-Darboux formula for the nth partial sum of the expansion of an arbitrary function in terms of orthogonal polynomials, the author tries to find a representation of the nth Fejér mean of the same expansion. The general formulas he obtains are simplified in the case of the classical orthogonal polynomials. *G. Szegö.*

Campbell, Robert. Séries de polynomes orthogonaux se prêtant au calcul explicite des sommes de Fejér. C. R. Acad. Sci. Paris **235**, 1092–1094 (1952).

In a preceding note [see the preceding review] the author introduced a method of calculating, in finite terms, the Fejér sums of the development of a function in series of certain orthogonal polynomials, comprising the usual ones. In the present note he determines all polynomials P_n to which this method applies. The necessary and sufficient conditions are stated in terms of the quantities involved in the recursion formulas

$$P_n(x)=l_n(x)P_{n-1}(x)-C_nP_{n-2}(x),$$
$$P_n(x)=f_n(x)P_{n-1}(x)-g_n(x)P'_{n-1}(x),$$

and are too lengthy to be reproduced here. *W. Rudin.*

Tandori, Károly. Über die Cesàrosche Summierbarkeit der orthogonalen Polynomreihen. Acta Math. Acad. Sci. Hungar. **3**, 73–82 (1952). (Russian summary)

Let $\{p_n(x)\}$ be the system of polynomials, orthonormal with respect to a non-negative weight function $w(x)$ over $\langle a,b\rangle$. Let $s_n(x)$ denote the nth partial sum of the Fourier series of $f(x)$ with respect to the $p_n(x)$, where $w^{1/2}f\in L^2$ over $\langle a,b\rangle$. The following theorem is proved: If $a\leq c<d\leq b$, and if the $p_n(x)$ are uniformly bounded in $\langle c,d\rangle$, then

$$(*)\qquad \frac{1}{n+1}\sum_0^n[s_k(x)-f(x)]^2\to0$$

p.p. in $\langle c,d\rangle$. If also $w(x)$ is uniformly bounded in $\langle c,d\rangle$, then (*) holds there at each point of continuity of $f(x)$, and uniformly in every closed interval of continuity contained in $\langle c,d\rangle$. This is a generalization of a familiar theorem of Hardy and Littlewood for trigonometrical Fourier series. An interesting consequence of the above theorem is: If $w(x)=0$ in a set of positive measure in $\langle c,d\rangle$, then the system $\{p_n(x)\}$ is not uniformly bounded in $\langle c,d\rangle$.

W. W. Rogosinski (Newcastle-on-Tyne).

Freud, Géza. Über die starke $(C,1)$-Summierbarkeit von orthogonalen Polynomreihen. Acta Math. Acad. Sci. Hungar. **3**, 83–88 (1952). (Russian summary)

The theorem of Tandori [see the preceding review] is localized and generalized as follows: If $w^{1/2}f\in L^2$ in $\langle a,b\rangle$, and if, for a fixed x, $a<x<b$,

$$(1)\qquad \sum_{k=0}^n p_k^2(x)=O(n)$$

and

$$(2)\qquad \int_x^{x+h}w(t)(f(t)-f(x))^2dx=o(|h|),$$

then

$$\frac{1}{n+1}\sum_{k=0}^n|s_k(x)-f(x)|\to0.$$

If $a\leq\alpha<\beta\leq b$, and if $w(x)\geq m>0$ in $\langle\alpha,\beta\rangle$, then (1) is satisfied uniformly in every closed subinterval of (α,β). If $0<m\leq w(x)\leq M$ and $f(x)$ is continuous in $\langle\alpha,\beta\rangle$, then $(n+1)^{-1}\sum_{k=0}^n s_k(x)\to f(x)$ uniformly in $\langle\alpha,\beta\rangle$. The corresponding results are familiar for trigonometrical Fourier series. *W. W. Rogosinski* (Newcastle-on-Tyne).

Freud, Géza. Über die Konvergenz orthogonaler Polynomreihen. Acta Math. Acad. Sci. Hungar. **3**, 89–98 (1952). (Russian summary)

Let $\rho(x)$ be a positive weight function in $[-1, 1]$ and let $\{p_n(x)\}$ denote the associated orthonormal polynomials. We assume that $(1-x^2)^a\rho(x)=O(1)$, $(1-x^2)^b p_n(x)=O(1)$, a and b positive. Let $\rho(x)f(x)$ be L-integrable and in a neighborhood of the end-points even $\rho(x)[f(x)]^2$ L-integrable. The author deals with the following generalization of the Dirichlet-Jordan and Hardy-Littlewood theorems: (1) Let $f(x)$ be of bounded variation in α, β where $-1<\alpha<\beta<1$, and continuous at x_0, $\alpha<x_0<\beta$. Then the expansion $\sum a_n p_n(x)$ of $f(x)$ is convergent for $x=x_0$ with the sum $f(x_0)$. If $f(x)$ is continuous in α, β, the convergence is uniform in each part of α, β. (2) Let $a_n=O(n^{-\gamma})$, $\gamma>0$, and $\log(|h|^{-1})(f(x_0+h)-f(x_0))\to 0$ as $h\to 0$, $-1<x_0<1$. Then the expansion converges at x_0 to the sum $f(x_0)$.

G. Szegö (Stanford, Calif.).

Alexits, Georg. Über den Annäherungsgrad der Orthogonalpolynomentwicklungen. Acta Math. Acad. Sci. Hungar. **3**, 43–48 (1952). (Russian summary)

Let $\{p_n(x)\}$ be the system of polynomials, orthonormal with respect to a non-negative weight function $w(x)$ over $\langle a, b\rangle$. Let $s_n(x)$ denote the nth partial sum of the Fourier series of $f(x)$ with respect to the $p_n(x)$ where $w^{1/2}f \in L^2$ over $\langle a, b\rangle$. Under the assumption that $w(x)$ and the $p_n(x)$ are (uniformly) bounded in $\langle a+\frac{1}{2}h, b-\frac{1}{2}h\rangle$, estimates are given for

$$\rho_n(a+h, b-h)=\max_{a+h\leq x\leq b-h}|f(x)-s_n(x)|, \quad h>0,$$

involving the continuity measure of the nth derivative $f^{(n)}(x)$. The order of magnitude of ρ_n is essentially the same as in the trigonometrical case. *W. W. Rogosinski.*

Special Functions

***Tricomi, Francesco. Funzioni ellittiche.** Seconda ed. Nicola Zanichelli Editore, Bologna, 1951. ix+343 pp. 4500 lire.

The first edition of this text appeared in 1937 and a somewhat augmented German edition, prepared by M. Krafft, was published in 1948 [Akademie Verlagsgesellschaft, Leipzig; these Rev. **10**, 532]. While including some additional material, the present edition is in general closer to the original Italian version than to the German edition.

Z. Nehari (St. Louis, Mo.).

Guinand, A. P. A note on the logarithmic derivative of the gamma function. Edinburgh Math. Notes **38**, 1–4 (1952).

The Fourier integral formula

$$\psi(x+1)-\log x=2\int_0^\infty[\psi(t+1)-\log t]\cos 2\pi xt\,dt,$$

$$\psi(z)=\frac{d}{dz}\log\Gamma(z),$$

and a summation formula connected with it, are examples of a general result of the author [J. London Math. Soc. **22**, 14–18 (1947); these Rev. **9**, 279]. In this note simple proofs are given for these two formulas. *A. Erdélyi.*

Ossicini, Alessandro. Funzione generatrice dei prodotti di due polinomi ultrasferici. Boll. Un. Mat. Ital. (3) **7**, 315–320 (1952).

The author sums $\sum_{n=0}^\infty n!s^nP_n^\lambda(x)P_n^\lambda(y)/\Gamma(2\lambda+n)$ and a few related series. *A. Erdélyi* (Pasadena, Calif.).

Ragab, Fouad M. An integral involving the product of a Bessel function and an E-function. Proc. Glasgow Math. Assoc. **1**, 8–9 (1952).

The author evaluates the integral

$$\int_0^\infty \lambda^{m-1}J_n(2\lambda)E(p;\alpha_r:q;\rho_s:x\lambda^{-2})d\lambda$$

as a linear combination of MacRobert's E-functions.

A. Erdélyi (Pasadena, Calif.).

Ragab, Fouad M. Generalisations of some integrals involving Bessel functions and E-functions. Proc. Glasgow Math. Assoc. **1**, 72–75 (1952).

The author evaluates

$$\int_0^\infty \lambda^{m-1}K_n(c\lambda)E(p;\alpha_r:q;\rho_s:x\lambda^{-c})d\lambda,$$

where $c=2^r$ and r is a positive integer, as an E-function, thus generalising a known result [MacRobert, Philos. Mag. (7) **31**, 254–260 (1941), equation (14); these Rev. **6**, 213]. He also considers a similar integral with J_n in place of K_n.

A. Erdélyi (Pasadena, Calif.).

Palamà, Giuseppe. Su di un limite inferiore della distanza di due zeri consecutivi di $H_n(x)$ e su di una limitazione di $H_n^2(x)-H_{n-1}(x)H_{n+1}(x)$. Boll. Un. Mat. Ital. (3) **7**, 311–315 (1952).

The author proves that

$$[H_n(x)]^2-H_{n-1}(x)H_{n+1}(x)<hn!\exp(x^2/2),$$
$$(n+1)[H_n(x)]^2-nH_{n-1}(x)H_{n+1}(x)<h(n+1)!\exp(x^2/2),$$

where $H_n(x)$ is the Hermite polynomial and $h=1.18034101$. He also obtains estimates for the distance of two consecutive zeros of a Hermite polynomial. *A. Erdélyi.*

MacRobert, T. M. On Neumann's formula for the Legendre functions. Proc. Glasgow Math. Assoc. **1**, 10–12 (1952).

The author makes some interesting comments on Neumann's formula

$$\frac{1}{2}\int_{-1}^1(z-x)^{-1}P_n(x)dx=Q_n(z), \quad n=0, 1, 2, \cdots.$$

He first shows that the insertion of a polynomial $f(x)$ of degree $\leq n$ in the integrand results in the appearance of the factor $f(z)$ on the right-hand side, and uses this remark to derive a number of integrals. He then evaluates the integral for general n and finds that its value is not, in general, $Q_n(z)$, but rather the function obtained by interpolating the sequence $\cdots, Q_2(z), Q_1(z), Q_0(z), Q_1(z), \cdots$ by means of E. T. Whittaker's cardinal series. The paper concludes with some similar integrals for associated Legendre functions for which the sum of order and degree is a non-negative integer.

A. Erdélyi (Pasadena, Calif.).

Martin, D. Integrals of Lommel's type for confluent hypergeometric functions. Proc. Glasgow Math. Assoc. 1, 28–31 (1952).

If $u_{k,m}(z)$ is a solution of the confluent hypergeometric equation

$$\frac{d^2u}{dz^2}+\left(-\frac{1}{4}+\frac{k}{z}+\frac{1-4m^2}{4z^2}\right)u=0,$$

then

$$\frac{d}{dz}[au'_{k,m}(az)u_{l,n}(bz)-bu_{k,m}(az)u'_{l,n}(bz)]$$
$$=\left[\frac{1}{4}(a^2-b^2)-\frac{ak-bl}{z}+\frac{m^2-n^2}{z^2}\right]u_{k,m}(az)u_{l,n}(bz).$$

The author uses this circumstance to evaluate several definite integrals involving Whittaker's functions $M_{k,m}(z)$ and $W_{k,m}(z)$. *A. Erdélyi* (Pasadena, Calif.).

MacRobert, T. M. Inequalities for a class of terminating generalised hypergeometric functions. Proc. Glasgow Math. Assoc. 1, 32–37 (1952).

The author proves by contour integration that for fixed α, β, γ, δ and unrestricted positive integer n there are constants M and m such that

$$|2^{2n}\,{}_3F_2(-n,\delta-n,\gamma-2n;\alpha-2n,\beta-2n;1)|<Mn^m,$$
$$|2^{-n}\,{}_3F_2(-n,\alpha,\beta;\gamma-\tfrac{1}{2}n,\delta-\tfrac{1}{2}n;1)|<Mn^m.$$

In the first of these inequalities it is assumed that α and β are not integers, in the second, that 2γ and 2δ are not integers. *A. Erdélyi* (Pasadena, Calif.).

MacRobert, T. M. Generalisations of some hypergeometric function transformations. Proc. Glasgow Math. Assoc. 1, 38–41 (1952).

The author expands $F(\alpha,\beta;\gamma;x)$ in powers of $x(1-x)$. When $2\gamma=\alpha+\beta+1$, the expansion reduces to one of Gauss' quadratic transformations. He gives a similar generalization of another one of Gauss' quadratic transformations, and two generalizations of Whipple's transformation. *A. Erdélyi.*

Nørlund, N. E. Séries hypergéométriques. Comm. Sém. Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.] Tome Supplémentaire, 169–172 (1952).

This is identical with Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] 21, no. 15 (1952); these Rev. 13, 752.

Meijer, C. S. Expansion theorems for the G-function. I. Nederl. Akad. Wetensch. Proc. Ser. A. 55 = Indagationes Math. 14, 369–379 (1952).

For the definition of the G-function see Nederl. Akad. Wetensch., Proc. 44, 1062–1070 (1941) [these Rev. 8, 155]. The object of the present paper is to prove some very general expansions for G-functions, and to show that a large number of transformation and expansion formulas for special functions are particular cases of the author's results. The present first part contains mostly introductory and preliminary matter, and also Theorem 1 which states the expansion

$$G_{pq}^{mn}\left(\lambda w\,\middle|\,{}^{a_r}_{b_s}\right)=\sum_{k=0}^{\infty}\frac{(\lambda-1)^k}{k!}G_{p+1,q+1}^{m,n+1}\left(w\,\middle|\,{}^{0,\,a_r}_{b_s,\,k}\right)$$

under precise conditions. *A. Erdélyi* (Pasadena, Calif.).

Chang, Chieh-Chien, Chu, Boa-Teh, and O'Brien, Vivian. An asymptotic expansion of the Whittaker function $W_{k,m}(z)$. J. Rational Mech. Anal. 2, 125–135 (1953).

The authors investigate $W_{k,m}(z)$ when $k=-k_0n+k_1$, $m=im_0n+m_1+o(n^{-1})$, $z=nz_0$, k_0, m_0, z_0 are positive real, and $n\to\infty$. They use the method of steepest descents except in the transition region where they use Langer's method. In an appendix they list in a tabular form all cases in which the asymptotic behavior of confluent hypergeometric functions has been determined. [The table is not complete and in part it is somewhat misleading. The important papers by Tricomi, and some papers by Lauwerier, are not mentioned. Some of the papers listed give the asymptotic behavior of $M_{k,m}(z)$ rather than that of $W_{k,m}(z)$.]

A. Erdélyi (Pasadena, Calif.).

Harmonic Functions, Potential Theory

Ninomiya, Nobuyuki. Sur une suite convergente de distributions de masses et leurs potentiels correspondants. Math. J. Okayama Univ. 2, 1–7 (1952).

Soit une suite $\{\mu_n\}$ de mesures de Radon positives portées par un compact fixe de R^m, convergeant vaguement vers μ; on sait [Deny, C. R. Acad. Sci. Paris 218, 497–499 (1944); ces Rev. 6, 228] qu'on peut en extraire une suite partielle dont le potentiel newtonien converge presque partout vers le potentiel engendré par μ. Dans cet énoncé peut-on remplacer "presque partout" par "quasi-partout" (i.e., "sauf sur un ensemble de capacité extérieure nulle")? L'auteur y parvient, mais seulement lorsque la suite $\{\mu_n\}$ satisfait à une condition supplémentaire qui, il est facile de le voir, est équivalente à la suivante: $\mu_n(e)\to\mu(e)$ pour tout ouvert $e\subset R^m$. Cette convergence, bien plus restrictive que la convergence vague, a fait l'objet d'une étude récente de J. Dieudonné [Anais Acad. Brasil. Ci. 23, 21–38 (1951); ces Rev. 13, 121]. *J. Deny* (Strasbourg).

Allen, A. C., and Kerr, E. The converse of Fatou's theorem. J. London Math. Soc. 28, 80–89 (1953).

A function $H(\xi,\eta)$ which is the difference of two nonnegative harmonic functions in the half-plane $\eta>0$ has a Poisson-Stieltjes representation

$$H(\xi,\eta)=D\eta+\int_{-\infty}^{\infty}\frac{\eta}{(t-\xi)^2+\eta^2}dg(t),$$

where D is a constant. The form of Fatou's theorem for which the authors find a converse is the following: If $g'(x)$ exists, then $H(\xi,\eta)\to\pi g'(x)$ as $(\xi,\eta)\to(x,0)$ along any curve in $\eta>0$ which is not tangent to $\eta=0$ at $(x,0)$. It is shown that if $g(t)$ is nondecreasing with $g(0)=0$, if α, β, γ, δ are constants with $0<\alpha<\beta<\pi$, $0<\gamma<\pi$, $-1<\delta<1$, and if $\lim_{s\to0+}s^\delta H_\alpha(s)=A$, $\lim_{s\to0+}s^\delta H_\beta(s)=B$, where

$$H_\alpha(s)=\int_{-\infty}^{\infty}\frac{s\sin\alpha}{t^2-2ts\cos\alpha+s^2}dg(t),$$

then

$$\lim_{s\to0+}s^\delta H_\gamma(s)=\{A\sin\delta(\beta-\gamma)+B\sin\delta(\gamma-\alpha)\}[\sin\delta(\beta-\alpha)]^{-1}.$$

This result is then used to evaluate $\lim_{t\to0+}t^{\delta-1}g(t)$ and $\lim_{t\to0+}-t^{\delta-1}g(-t)$. *A. J. Lohwater* (Ann Arbor, Mich.).

Kerr, E. A note on positive harmonic functions. J. London Math. Soc. 28, 89–94 (1953).

The author extends a result of the paper reviewed above [cf. the review for notation] to the case of tangential approach. The following extension is typical: Let $H(\xi, \eta)$ be harmonic and positive in the half-plane $\eta>0$; if $H(\xi, \eta)\to A$ as $(\xi, \eta)\to(0, 0)$ along a tangential path in $\eta>0$, $\xi\geqq 0$, and if $H(\xi, \eta)\to B$ as $(\xi, \eta)\to(0, 0)$ along a tangential path in $\eta>0$, $\xi\leqq 0$, then

$$\lim_{t\to 0+} t^{-1}g(t)=A/\pi \text{ and } \lim_{t\to 0+} -t^{-1}g(-t)=B/\pi.$$

A. J. Lohwater (Ann Arbor, Mich.).

Kuroda, Tadashi. Notes on an open Riemann surface. II. Kōdai Math. Sem. Rep. 1952, 36–38 (1952).

[For part I see same Rep. **1951**, 61–63; these Rev. **13**, 735.] The author establishes a number of known results concerning harmonic functions with bounded Dirichlet integral due to Bader and Parreau, Mori, and Nevanlinna.

M. H. Heins (Paris).

Ullemar, Leo. Über die Existenz der automorphen Funktionen mit beschränktem Dirichletintegral. Ark. Mat. 2, 87–97 (1952).

The author introduces a variant of the classical harmonic measure, which he terms the second harmonic measure. It is defined as follows: Let G denote a simply connected Jordan region with boundary Γ, let E denote a closed subset of Γ, and $E'=\Gamma-E$. By $\Omega(z_0, E, G)$, the second harmonic measure of E at $z_0 \varepsilon G$, is meant sup $u(z_0)$, where u is harmonic in G and satisfies $\lim\sup_{z\to E'} u\leqq 0$ and $D_G(u)\leqq\pi$. A number of properties of Ω are developed and with its aid a necessary and sufficient condition is given for the existence of a nonconstant analytic function which is (1) automorphic with respect to a given symmetric Hauptkreis group of genus zero without elliptic transformations and (2) has a finite Dirichlet integral over a fundamental domain of the group.

M. H. Heins (Paris).

Lax, Peter D. On the existence of Green's function. Proc. Amer. Math. Soc. 3, 526–531 (1952).

Breve dimostrazione della esistenza della funzione di Green per il Δ_2 in r variabili, basata sul teorema di Hahn-Banach relativo al prolungamento dei funzionali lineari. In questo stesso ordine di idee confronta P. R. Garabedian [Trans. Amer. Math. Soc. **69**, 392–415 (1950); questi Rev. **12**, 492] e C. Miranda [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 55–59 (1947); questi Rev. **9**, 238].

C. Miranda (Napoli).

Sharma, A. On an application of a method of Shohat to a problem of Lukacs. Ganita 2, 9–22 (1951).

The following analogue of a theorem of Lukács is proved. Let $U(x_0, x_1, \cdots, x_v)$ be a "polyharmonic" function of order n within the hypersphere $S_\rho(M)$ with center M and radius ρ, i.e., $\Delta^{(n)}U=0$. We denote by $\mu(M; r)$ the mean-value $[\sigma_n(r)]^{-1}\int U d\sigma$ where the integration is extended over the surface $S_r(M)$ having the surface area $\sigma_n(r)$. If A and B denote the minimum and maximum of $\mu(M; r)$, $0\leqq r\leqq\rho$, we have

$$A+\frac{B-A}{\tau}\leqq v\rho^{-v}\int_0^\rho \mu(M; r)r^{v-1}dr\leqq B-\frac{B-A}{\tau_n}$$

where

$$\tau_n=\begin{cases}(m+1)(2m/v+1), & n=2m+1,\\ (m+1)(2m/v+2/v+1), & n=2m+2.\end{cases}$$

G. Szegö (Stanford, Calif.).

Egloff, Werner. Eine mit der Theorie der Kugelverbiegungen zusammenhängende Eigenwertaufgabe der Potentialtheorie. Math. Nachr. 8, 99–122 (1952).

Domains D on the surface of a sphere are to be determined, for which there exists an infinitesimal isometric deformation moving all points on the boundary of D parallel to the equatorial plane of the sphere. Liebmann in 1920 showed the existence of a discrete set of circular domains D on the sphere with this property. In general, one can reduce the problem by stereographic projection from the south-pole to a "third boundary-value problem" for the potential equation in the image domain D' of D. [A linear combination of the function and its first derivatives has to vanish on the boundary.] The solution of this problem leads to a homogeneous Fredholm equation. The author discusses the existance of solutions of this problem for the special case, where D' is an ellipse. Using a Fourier series expansion the integral equation goes over into a rather complicated system of difference equations for the coefficients. The determination of eigenvalues is achieved by means of continued fractions. A numerical scheme is discussed for finding an ellipse D' of given shape for which the problem has a solution.

F. John (New York, N. Y.).

Bertolini, F. Sulla capacità di un condensatore sferico. Nuovo Cimento (9) 9, 852–854 (1952).

The capacity c_0 of a condenser C_0 formed from two concentric spherical shells of radii r and R $(r<R)$ is given by $c_0=rR/(R-r)$. The author considers the modified condenser C obtained by cutting a circular hole of angular radius δ in the outer shell. The capacity c of C is shown to satisfy the two-sided inequality

$$\frac{(r/R)^2\sin^4(\frac{1}{2}\delta)}{1-(r/R)\cos\delta}\leqq\frac{c_0-c}{c_0}\leqq(r/R)\sin^2(\tfrac{1}{2}\delta).$$

J. W. Green (Los Angeles, Calif.).

Differential Equations

Hukuhara, Masuo. Sur les propriétés de la famille des courbes intégrales d'un système différentiel ordinaire. Proc. Japan Acad. 25, 151–153 (1949).

If $f_j(x, y_k)$ $(j, k=1, \cdots, n)$ are continuous and bounded, then a solution of $y_j'=f_j(x, y_k)$ through P is locally unique at P if and only if there exists a neighborhood of P such that no solution curve through P has in that neighborhood either a right bifurcation point to the right of or at P or a left bifurcation point to the left of or at P.

F. A. Ficken (Knoxville, Tenn.).

Burton, L. P., and Whyburn, William M. Minimax solutions of ordinary differential systems. Proc. Amer. Math. Soc. 3, 794–803 (1952).

Se le funzioni $f_i(x, y_1, \cdots, y_n)$, $i=1, \cdots, n$, sono continue e soddisfano per $x\geqq x_0$ a convenienti condizioni di monotonia rispetto alle variabili y_j, il sistema differenziale $y_i'=f_i(x, y_1, \cdots, y_n)$, $y_i(x_0)=y_i^0$, ammette un solo integrale $Y_i(x)$ tale che per ogni altro integrale $y_i(x)$ riesca, per $x_0\leqq x\leqq x_0+h$, $y_i\leqq Y_i(x)$ per $i\leqq k$, $Y_i(x)\leqq y_i(x)$ per $k<i$, il valore di k restando fissato dalle ipotesi fatte sulle f_i.

C. Miranda (Napoli).

Markus, Lawrence. On completeness of invariant measures defined by differential equations. J. Math. Pures Appl. (9) 31, 341–353 (1952).

Consider the system of differential equations

$$(*)\qquad \frac{dx}{dt}=f(x,y),\quad \frac{dy}{dt}=g(x,y)$$

where f and g are of class $C^{(n)}$ ($n\geqq 1$) and $f^2+g^2>0$ in a simply connected region R of the plane. Let $\mu(x,y)$ (an integrating factor) be non-negative and of class C^m ($m\geqq 1$) in R such that μ vanishes on no open set of R and

$$\frac{\partial}{\partial x}(\mu f)+\frac{\partial}{\partial y}(\mu g)=0$$

in R. Then $m_\mu(\Lambda)=\int\int_\Lambda \mu(x,y)dxdy$ is a measure defined for Λ in the class L of Lebesgue measurable sets and m_μ is invariant under the flow defined by (*). It is shown that: (1) $n=1$ and $m=A$ (i.e., μ analytic) imply that m_μ is complete; that is, $m_\mu(\Lambda)=0$ if and only if $m_L(\Lambda)=0$, $\Lambda\in L$; (2) it is possible to construct an example (*) corresponding to any simply connected R such that f, g, and μ are of class C^∞ and m_μ is not complete, for there exists $Z\in L$ such that $m_\mu(Z)=0$ but $m_L(Z)>0$. *G. A. Hedlund.*

Haas, Felix. A theorem about characteristics of differential equations on closed manifolds. Proc. Nat. Acad. Sci. U. S. A. 38, 1044–1047 (1952).

Let V denote a vector field which satisfies a Lipschitz condition on a closed orientable 2-manifold M. Let C^+ denote a positive semi-characteristic of the linear differential equation belonging to V and let $\bar{C}$ denote the set of ω-limit points of C^+. The author sketches a proof of the following theorem which is to be proved in detail elsewhere: If V has at most a denumerable number of singular points and if there exists a C^+ such that $\bar{C}$ contains no singular points, then either M is a torus and V has no singular points or $\bar{C}$ is nowhere dense on M. *C. J. Titus.*

Gallissot, François. Transformations infinitésimales et intégration des équations différentielles de la mécanique. C. R. Acad. Sci. Paris 235, 1599–1600 (1952).

This paper is a continuation of a previous one [same C. R. 235, 1277–1278 (1952); these Rev. 14, 378]. It is concerned with the study, by means of operators recently introduced by H. Cartan, of the integration of the differential system Σ associated to a form Ω of degree 2 and rank $2n$ in a manifold V_{2n+1} of dimension $2n+1$, in the case $d\Omega\neq 0$. It is stated that to an infinitesimal transformation x corresponds a Pfaffian form π which vanishes on the integral curves of Σ, and that, conversely, to such a form corresponds an infinitesimal transformation modulo the characteristic field E associated to Σ. If to X corresponds an infinitesimal transformation of Σ and to $\bar{x}$ a first integral of Σ, then to $[X,\bar{x}]\neq 0$ corresponds to the new first integral $i(X\wedge\bar{x})\Omega$ of Σ. There are $2n$ infinitesimal transformations $x^1,\cdots,x^{2n}$, such that, if $\pi^k=i(x^k)\Omega$, $k=1,\cdots,2n$, then the system of Pfaffian forms $\pi^{r+1},\cdots,\pi^{2n}$ is completely integrable and the r forms $\pi^1,\cdots,\pi^r$ are invariant. *S. Chern* (Chicago, Ill.).

Erugin, N. P. Construction of the whole set of systems of differential equations having a given integral curve. Akad. Nauk SSSR. Prikl. Mat. Meh. 16, 659–670 (1952). (Russian)

Questions of the following nature are dealt with: Under what conditions does the system

$$(1)\qquad \dot{x}=P(x,y),\quad \dot{y}=Q(x,y)$$

have $w(x,y)=0$ as trajectory. The answer is that the system must have the form

$$\dot{x}=F_1(w,x,y)-\frac{\partial w}{\partial y}M(x,y),\quad \dot{y}=F_2(w,x,y)+\frac{\partial w}{\partial x}M(x,y),$$

where $F_i(0,x,y)=0$, and M is an arbitrary function. If (1) is to have the trajectories $w_1(x,y)=0$, $w_2(x,y)=0$, then it must be of the form

$$\dot{x}=\frac{\partial w_2}{\partial y}F_1-\frac{\partial w_1}{\partial y}F_2,\quad \dot{y}=-\frac{\partial w_2}{\partial x}F_1+\frac{\partial w_1}{\partial x}F_2,$$

where the F_i behave as before. Various special cases are considered. The extension to a system

$$\dot{x}=P(x,y,z),\quad \dot{y}=Q(x,y,z),\quad \dot{z}=R(x,y,z)$$

with assigned trajectory $D_1(x,y,z)=0$, $D_2(x,y,z)=0$ is likewise taken up and it is shown that then

$$P=p(D_1,D_2,x,y,z)+\frac{\partial(D_1,D_2)}{\partial(y,z)}M(x,y,z),$$

with $p(0,0,x,y,z)=0$, and similar formulas for Q and R. *S. Lefschetz* (Princeton, N. J.).

✱Dubošin, G. N. Osnovy teorii ustoičivosti dviženiya. [Foundations of the theory of stability of motion.] Izdat. Moskov. Univ., Moscow, 1952. 318 pp. 8.30 rubles.

In Lyapunov's classical mémoire: Problème général de la stabilité du mouvement [Ann. Fac. Sci. Univ. Toulouse (2) 9, 203–474 (1907) = Annals of Mathematics Studies, no. 17, Princeton, 1947; these Rev. 9, 34], the theme is the stability of the origin for a system (*) $\dot{x}=X(x,t)$, where $\dot{x}$ and X are n-vectors, the components X_i of X being holomorphic in x near the origin with coefficients continuous and bounded or also periodic in t. Lyapunov uses on the one hand direct analytical expressions for the solutions of (*)—this is his first method—on the other hand his second and indirect method by means of the solutions of certain partial differential equations. Roughly speaking, the stability or instability is established by obtaining an oval surrounding the origin which contracts or expands upon following the trajectories. This second method has the advantage that, in principle at least, it does not require analyticity and dispenses with the solution of the characteristic equation. The mémoire of Lyapunov deals also at the beginning with his notion of characteristic number of a real function $f(t)$ as $t\rightarrow+\infty$, a notion which does not appear to have been as yet exploited as it should be.

The present volume is primarily a careful presentation and reworking of Lyapunov's second method with many examples carried out in detail and from that point of view it is most praiseworthy. However, since Lyapunov, a number of additions have been made to his theory, mostly by Soviet authors (notably Persidskiĭ and Malkin), and to these contributions the author makes merely cursory references. Even more glaring is the total omission of any mention of Poincaré whose work had anticipated Lyapunov's in a number of places by some years. Only a bare mention is made of Lyapunov's characteristic numbers. *S. Lefschetz* (Princeton, N. J.).

Dubošin, G. N. A stability problem for constantly acting disturbances. Vestnik Moskov. Univ. Ser. Fiz.-Mat. Estest. Nauk 1952, no. 2, 35–40 (1952). (Russian)

Given the system $\dot{z}=Z(t;z;\mu)$, where z, Z are k-vectors and μ is an r-vector, it is supposed that $\dot{z}^*=z(t;z^*;\mu^0)$ has

a solution $z^*(t; z^0; \mu^0)$. The question of the stability of this solution is discussed. By introducing new variables

$$x_i = z_i - z_i^*, \quad i \leqq k, \quad x_{k+j} = \mu_j - \mu_j^0$$
$$X_i = Z_i(t; z; \mu) - Z_i(t; z^*; \mu^0), \quad i \leqq k,$$
$$X_{k+j} = 0,$$

the question is reduced to the discussion by standard (?) methods of the ordinary system $\dot{x} = X(t; x)$, where x, X have for components x_i, X_i. *S. Lefschetz* (Princeton, N. J.).

Papuš, P. N. On finding regular semi-stable limit cycles. Uspehi Matem. Nauk (N.S.) 7, no. 4(50), 165–168 (1952). (Russian)

In order that the closed trajectory L of $dy/dx = f(x, y)$ be a semistable regular limit cycle it is necessary that $\Phi = ff_x - f_y = 0$, $\Phi_x + f\Phi_y = 0$ on L. *J. L. Massera.*

Kulikov, N. K. On the determination of the limit of the general solution of a nonlinear differential equation of the first order. Akad. Nauk SSSR. Prikl. Mat. Meh. 16, 729–734 (1952). (Russian)

The author considers the equation

$$\dot{x} + F(x) = M \sin pt, \quad 0 < h < F'(x) = F_x < \infty,$$

assumes that the general solution has a limiting solution x_∞ as $t \to \infty$, and proves that the limiting solution is given by the relation

$$\frac{F_\infty}{F_{x_\infty}} = \frac{M(F_{x_\infty} \sin p\tau - p \cos p\tau)}{p^2 + F^2_{x_\infty}}$$

where F_∞, F_{x_∞}, are F, F_x taken at x_∞. *S. Lefschetz.*

Manacorda, Tristano. Sul comportamento asintotico degli integrali di una classe di sistemi di equazioni differenziali lineari non omogenei. Boll. Un. Mat. Ital. (3) 7, 281–284 (1952).

Assuming that the coefficients of the system of differential equations

$$\dot{x}_h = \sum_{k=1}^{n} a_{hk}(t) x_k + \eta_h(t) \quad (h = 1, \cdots, n)$$

satisfy the conditions: (1) $a_{hk}(t)$, $\eta_h(t)$ continuous for $t_0 \leqq t < \infty$, (2) $\int^\infty a_{hh} dt$ convergent, (3) $\int^\infty dt \left| \int_t^\infty a_{hh} ds \right| < \infty$, (4) $\limsup_{t\to\infty} |a_{hk}| < \infty$, (5) $\int^\infty (\sum_{k=1}^n a_{hk}(t) \int^t \eta_k(s) ds) dt$ convergent, (6) $\limsup_{t\to\infty} |\sum_{k=1}^n a_{hk}(t) \int^t \eta_k(s) ds| < \infty$, it is proved by the method of successive approximations that every solution of the system tends to a solution of the system $\dot{x}_h = \eta_h(t)$, as $t \to \infty$. *W. Wasow.*

Blaquière, A. Les oscillateurs non linéaires et le diagramme de Nyquist. J. Phys. Radium (8) 13, 527–540 (1952).

The author reviews the ordinary theory of linear oscillators. He then attempts to treat certain nonlinear oscillators by means of adaptations of the concepts and methods of the linear theory. In the opinion of the reviewer the arguments advanced are not convincing, and the value of the results obtained is uncertain. *L. A. MacColl.*

Rubinowicz, A. Sommerfeld's polynomial method simplified. Proc. Phys. Soc. Sect. A. 63, 766–771 (1950).

In earlier papers [Nederl. Akad. Wetensch., Proc. 52, 351–362 = Indagationes Math. 11, 125–136 (1949); Proc. Phys. Soc. Sect. A. 62, 736–738 (1949); these Rev. 11, 178, 597] the author has shown that solutions of the Sturm-Liouville equation obtainable by Sommerfeld's polynomial method can be written in terms of hypergeometric and confluent hypergeometric functions. Here he finds formulas for the constants determining the eigenvalues and eigenfunctions. The method is used to solve the associated Legendre equation and the radial equation for the non-relativistic one electron atom. *T. E. Hull.*

Stevenson, A. F., and Bassali, W. A. On the possible forms of differential equation which can be factorized by the Schrödinger-Infeld method. Canadian J. Math. 4, 385–395 (1952).

The factorization method enables one to find the eigenvalues, the normalized eigenfunctions, the recurrence relations, and certain matrix elements associated with a number of differential equations of the form (A) $y'' + r(x, m)y + \lambda y = 0$. This equation must first be "factorized" and the important problem of finding what types of equations can be factorized can be reduced to one of solving a certain differential-difference equation. The authors find all types under the one assumption that (A) can be transformed so that its coefficients are rational functions. Their list includes one essentially new type as well as those found earlier [Infeld and Hull, Rev. Modern Physics 23, 21–68 (1951); these Rev. 13, 239] under a more restrictive assumption. *T. E. Hull.*

Satoh, Tunezo. Some complex boundary valued problem and its application to Kapur-Peierls' dispersion formula. J. Phys. Soc. Japan 7, 474–481 (1952).

Green's functions, some integral equation theory, and an approximation procedure are used to solve the boundary value problem:

$$y''(x) - q(x)y + \lambda y = 0, \quad 0 \leqq x \leqq 1,$$
$$y(0) = 0, \quad y'(1) - iky(1) = I \exp(-ik).$$

The results are equivalent to those of Kapur and Peierls [Proc. Roy. Soc. London. Ser. A. 166, 277–295 (1938)] who used their results to obtain a "dispersion" formula for nuclear reactions. *T. E. Hull* (Vancouver, B. C.).

Miller, K. S. Self-adjoint differential systems. Quart. J. Math., Oxford Ser. (2) 3, 175–178 (1952).

The author considers the linear system

$$(1) \qquad Lu \equiv p_i(x) u^{(n-i)}, \quad U_\alpha(u) = V_\alpha(u) + Z_\alpha(u),$$

where $V_\alpha(u) = A_{\alpha i} u^{(i)}(a)$, $Z_\alpha(u) = B_{\alpha i} u^{(i)}(b)$ ($\alpha = 1, \cdots, n$; i summed over $0, 1, \cdots, n-1$). He supposes that the system $Lu = 0$, $U_\alpha(u) = 0$ is incompatible. If $\{\phi_\beta(x)\}_1^n$ are n linearly independent solutions of $Lu = 0$, one writes $U_{\alpha\beta} = U_\alpha(\phi_\beta)$ and similarly $\Delta_{\alpha\beta} = V_{\alpha\beta} - Z_{\alpha\beta}$. Next, define the matrices $D = \|U_{\alpha\beta}\|$, $\Delta = \|\Delta_{\alpha\beta}\|$, and $C = \|c_{ij}\|$, where

$$\|\phi_i^*(x)\| = \|c_{ij}\| \cdot \|\phi_j(x)\|,$$

and $\{\phi_i^*(x)\}_1^n$ is the adjoint system of solutions of $L^+u = (-1)^n Lu = 0$, Lu being assumed formally self-adjoint. Using the Green's function for the original system he proves that a necessary and sufficient condition that the system (1) be self-adjoint is that $\Delta CD'$ be symmetric (skew-symmetric) if L is a formally self-adjoint differential operator of even (odd) order. D' is the transpose of D.

W. Leighton (St. Louis, Mo.).

Putnam, C. R. On the unboundedness of the essential spectrum. Amer. J. Math. 74, 578–586 (1952).

Assume that the differential equation $x'' + (\lambda - f)x = 0$ is of the limit-point type at ∞, where f is a real-valued continuous function on $0 \leqq t < \infty$. Let S' denote the set of cluster points of the spectrum corresponding to this differ-

ential equation and a homogeneous boundary condition at zero, and let $m(\lambda)=\min|\lambda-\mu|$, for $\mu \varepsilon S'$. It is proved that if f is bounded from below, then either S' is empty or S' is unbounded from above, and in the latter case $m(\lambda)=O(\lambda^{\frac{1}{2}})$, as $\lambda\to\infty$. Certain other consequences of the method are pointed out. *E. A. Coddington* (Los Angeles, Calif.).

Hartman, Philip, and Wintner, Aurel. On perturbations of the continuous spectrum of the harmonic oscillator. Amer. J. Math. 74, 79–85 (1952).

Consider the differential equation (D): $x''+(\lambda+f)x=0$, and the boundary condition (B): $x(0)\cos\alpha+x'(0)\sin\alpha=0$, where f is real, continuous on $0\leqq t<\infty$, and $0\leqq\alpha<\pi$. It is shown that if $\lim\inf T^{-1}\int_0^T|f(t)|dt=0$ $(T\to+\infty)$, then (D) is of limit-point type at ∞, that is, (D) and (B) determine a self-adjoint operator in the Hilbert space $L^2(0,\infty)$. Under the same assumption on f, it is proved that the set of cluster points of the spectrum of this operator contains the interval $0\leqq\lambda<\infty$. (There are two slight errors, which however do not affect the proofs: (i) after formula (6) the set S_n' should be the derived set of S_n, instead of the closure as stated; (ii) after formula (28) it is stated that the sum of a self-adjoint and a bounded operator is always self-adjoint; the bounded operator must be self-adjoint.)
E. A. Coddington (Los Angeles, Calif.).

Naimark, M. A. On the spectrum of singular non-self-adjoint differential operators of the second order. Doklady Akad. Nauk SSSR (N.S.) 85, 41–44 (1952). (Russian)

The differential expression $l(y)=-y''+p(x)y$, $0\leqq x<\infty$, where $p(x)$ is a complex-valued function of x integrable over every interval $[0,a]$, $a>0$. The author defines $\mathfrak{D}$ as the set of functions $y\,\varepsilon\,\mathfrak{L}^2(0,\infty)$ such that y and y' are absolutely continuous over $[0,a]$ for all $a>0$ and such that $l(y)\,\varepsilon\,\mathfrak{L}^2(0,\infty)$. Let θ be a complex number. Let $\mathfrak{D}_\theta$ denote the set of functions $y\,\varepsilon\,\mathfrak{D}$ such that $y'(0)-\theta y(0)=0$. Denote by L_θ the operator defined for $y\,\varepsilon\,D_\theta$ by $L_\theta y=l(y)$. If the equation $l(y)=\lambda y$ has a solution $y\,\varepsilon\,\mathfrak{D}_\theta$, then λ is said to be in the spectrum of L_θ. The author then considers a number of cases. In the simplest, $p(x)\,\varepsilon\,\mathfrak{L}(0,\infty)$. For this case he states that L_θ has a continuous spectrum for positive real λ and discrete otherwise. For λ not in the spectrum the resolvent $(L_\theta-\lambda I)^{-1}$ is an integral operator $K(x,y,\lambda)$ with

$$\int_0^\infty|K(x,y,\lambda)|^2dx+\int_0^\infty|K(x,y,\lambda)|^2dy<\infty.$$

Many other results are stated. *N. Levinson.*

Molčanov, A. M. Criteria of discreteness of the spectrum of a differential equation of the second order. Doklady Akad. Nauk SSSR (N.S.) 83, 17–18 (1952). (Russian)

In euclidean n-space the equation

$$(*)\qquad -\left(\frac{\partial^2\psi}{\partial x_1^2}+\cdots+\frac{\partial^2\psi}{\partial x_n^2}\right)+q(x_1,\cdots,x_n)\psi=\lambda\psi$$

is considered, where q is defined for all $(x_1,\cdots,x_n)$ and is bounded below. For such equations necessary and sufficient conditions are stated for the discreteness of the spectrum. For $n=1$ the spectrum for the equation $-\psi''+q\psi=\lambda\psi$ is discrete if and only if for any fixed $\delta>0$ it is true that $\int_N^{N+\delta}q(x)dx\to+\infty$ as $N\to+\infty$ or $N\to-\infty$. For the cases $n>1$ new difficulties arise, and the criterion for discreteness becomes more complicated, although analogous to the condition given for $n=1$. Various criteria for the non-discreteness of the spectrum for the problem

$$-(\partial^2\psi/\partial x_1^2+\cdots+\partial^2\psi/\partial x_n^2)=\lambda\psi$$

on a domain G, $\psi=0$ on the boundary of G, are given. A result for difference equations is also stated: the equation

$$\begin{aligned}-\textstyle\sum_{k=1}^n[\psi(i_1,\cdots,i_k+1,\cdots,i_n)-2\psi(i_1,\cdots,i_k,\cdots,i_n)\\+\psi(i_1,\cdots,i_k-1,\cdots,i_n)]\\+q(i_1,\cdots,i_n)\psi(i_1,\cdots,i_n)=\lambda\psi(i_1,\cdots,i_n)\end{aligned}$$

possesses a discrete spectrum if and only if $q(i_1,\cdots,i_n)\to+\infty$ as $|i_1|+\cdots+|i_n|\to+\infty$. (Other recent works concerned with the discreteness of the spectrum of $-\Delta\psi+q\psi=\lambda\psi$ are (1) Friedrichs, Studies and Essays Presented to R. Courant..., Interscience, New York, 1948, pp. 145–160; these Rev. 9, 353; (2) Rellich, ibid. pp. 329–344; these Rev. 9, 355; (3) Brownell, Ann. of Math. (2) 54, 554–594 (1951); these Rev. 13, 847.) *E. A. Coddington* (Los Angeles, Calif.).

Višik, M. I. On general boundary problems for elliptic differential equations. Trudy Moskov. Mat. Obšč. 1, 187–246 (1952). (Russian)

Let L be an elliptic differential operator for which the equation $Lu=h$ in a region D with u vanishing at the boundary has a unique solution which depends continuously on h, considered as an element of $\mathfrak{L}_2(D)=\mathfrak{L}_2$. Two operators correspond to L, a maximal one L_1 whose domain of definition $\mathfrak{D}_1$ consists of all f in $\mathfrak{L}_2$ for which Lf is also in $\mathfrak{L}_2$, and a minimal one $L_0\subset L_1$ whose domain $\mathfrak{D}_0$ consists, roughly speaking, of those elements of $\mathfrak{D}_1$ which vanish at the boundary of D. The author establishes a connection between certain intermediate linear operators A, $L_0\subset A\subset L_1$, and homogeneous boundary conditions for the equation $Lu=h$. The main results have been announced before [Doklady Akad. Nauk SSSR (N.S.) 65, 433–436, 785–788 (1949); ibid. 77, 373–375, 553–555 (1951); these Rev. 11, 38, 39; 12, 830].
L. Gårding (Lund).

Browder, Felix E. The Dirichlet and vibration problems for linear elliptic differential equations of arbitrary order. Proc. Nat. Acad. Sci. U. S. A. 38, 741–747 (1952).

The results previously given by the author [same Proc. 38, 230–235 (1952); these Rev. 14, 174] are extended in various directions. (See the cited review for the notation used.) In the previous note the elliptic operators K considered were assumed to be "positive", i.e., there should exist a constant c such that $(f,f)\geqq c(f,f)_m$ for all f in C_c^∞. It is shown now that ellipticity of K implies that $K+d$ is positive for some real number d. In a different form (shown here to be equivalent) this basic result had been announced previously by Gårding [C. R. Acad. Sci. Paris 233, 1554–1556 (1951); these Rev. 14, 174]. Gårding's proof for the case of constant coefficients forms the starting point for the proof given here for general elliptic K with variable coefficients. A second set of theorems concerns eigenvalue problems. The space of complex functions u, which have "zero boundary values" in the sense defined in the previous note, is denoted by $\hat{H}$. Given an operator B of lower order than K, the equation $Ku=\lambda Bu$ for u in $\hat{H}$ defines the eigenfunctions and eigenvalues of K with respect to B. It is shown that for elliptic K and positive self-adjoint B the eigenvalues form a discrete set and have finite multiplicity. In case K is also self-adjoint, the eigenfunctions are complete. Extending a result in the small of the reviewer, the author proves the existence of a fundamental solution belonging to an elliptic K in any bounded domain, for which the solution of the Dirichlet problem is unique. Finally, some theorems are

given for unbounded domains with exterior points. It turns out that the solution of the Dirichlet problem is unique for a positive self-adjoint elliptic K, and that it exists for all boundary value functions g, which are "permissible" in a certain sense. More generally, an alternative is derived for K which differs only by lower order terms from a positive self-adjoint operator and satisfies certain conditions at infinity. *F. John* (New York, N. Y.).

Kalandiya, A. I. Remark on the uniqueness of solution of a fundamental boundary problem for a class of elliptic equations. Soobščeniya Akad. Nauk Gruzin. SSR **12**, 321–325 (1951). (Russian)

The author proves the following theorem by means of suitable transformations of Green's identity. If T is a sufficiently small plane domain, L is its boundary, the real-valued function $u(x, y)$ is continuous on $T+L$ together with $\partial^{k+m}u/\partial z^k\partial\bar{z}^m$, $k\leqq n$, $m\leqq n$, where $\partial/\partial z=\frac{1}{2}(\partial/\partial x-i\partial/\partial y)$, $\partial/\partial\bar{z}=\frac{1}{2}(\partial/\partial x+i\partial/\partial y)$, and further

$$\Delta^n u+b_1\Delta^{n-1}u+\cdots+b_n u=0 \quad \text{in } T,$$

$(\Delta=\partial^2/\partial x^2+\partial^2/\partial y^2)$ $n\geqq 1$, $b_1, b_2, \cdots, b_n$ real constants,

$$u=0, \frac{du}{d\nu}=0, \cdots, \frac{d^{n-1}u}{d\nu^{n-1}}=0 \quad \text{on } L,$$

(ν the outer normal), then u is identically zero. Using a theorem of I. N. Vekua [New methods for the solution of elliptic equations, Moscow-Leningrad, 1948; these Rev. **11**, 598] the author is thus able to assert the unique solvability of the fundamental boundary-value problem (u, $du/d\nu$, $\cdots$, $d^{n-1}u/d\nu^{n-1}$ prescribed on the boundary L) for the partial differential equation under consideration. A uniqueness theorem (for sufficiently small T) for the fundamental boundary value problem for the second-order equation with variable coefficients

$$\Delta u+a\frac{\partial u}{\partial x}+b\frac{\partial u}{\partial y}+cu=0,$$

was given by E. Picard [J. Math. Pures Appl. (4) **6**, 145–210, 231 (1890)], and for the fourth-order equation with variable coefficients by Bremekamp [Nederl. Akad. Wetensch., Proc. **45**, 546–552 (1942); these Rev. **5**, 242] using methods which according to the author are not directly applicable to the present situation. *J. B. Diaz.*

Grosberg, Yu. I. On the application of B. G. Galerkin's method to problems with nonhomogeneous boundary conditions. Doklady Akad. Nauk SSSR (N.S.) **85**, 473–476 (1952). (Russian)

It is proved that in the case of elliptic partial differential equations subject to boundary conditions of the second and third kinds, the "coordinate functions" to be chosen in the application of Galerkin's method need not be required to fulfill the boundary conditions of the given boundary-value problem. Specifically, the following theorem is proved (Ω is a bounded open set with piece-wise smooth boundary Γ, and $u_1(x), \cdots, u_k(x), \cdots, x=(x_1, \cdots, x_n)\in\Omega$, is a sequence of "coordinate functions", twice continuously differentiable on $\Omega+\Gamma$, supposed to be complete with respect to the metric defined by $\|u\|^2=\int_\Omega[\sum_{k=1}^n(\partial u/\partial x_k)^2+u^2]d\Omega$): If the boundary value problem

$$L[u]\equiv\sum_{k,j=1}^n\frac{\partial}{\partial x_k}\left(a_{jk}\frac{\partial u}{\partial x_j}\right)-\sum_{k=1}^n b_k\frac{\partial u}{\partial x_k}+cu=f(x), \quad x\in\Omega;$$

$$\Lambda[u]\equiv\sum_{k,j=1}^n\cos(\nu, x_k)a_{jk}\frac{\partial u}{\partial x_j}-\gamma u=\varphi(s), \quad s\in\Gamma$$

(where a_{jk} and their first derivatives, plus b_k and c, are all continuous on $\Omega+\Gamma$ and γ is bounded on Γ; and the quadratic form $\sum_{k,j=1}^n a_{jk}\xi_j\eta_k$ is positive definite for each point of $\Omega+\Gamma$) has a unique solution $u_0(x)$, then, for all sufficiently large positive integers m, the system of "Galerkin equations"

$$\sum_{p=1}^m\left\{\int_\Omega L[u_p]u_q d\Omega+\int_\Gamma\Lambda[u_p]u_q d\Gamma\right\}c_p=\int_\Omega fu_q d\Omega+\int_\Gamma\varphi u_q d\Gamma,$$

$q=1, 2, \cdots, m$, has a non-zero determinant, and the corresponding sequence of functions

$$u^m(x)=\sum_{p=1}^m c_p u_p(x)$$

converges "in the mean" (i.e., in the sense of the metric defined above) to the solution $u_0(x)$. *J. B. Diaz.*

Miles, John W. On solutions to the wave equation in hyperbolic space. J. Appl. Phys. **23**, 1400–1401 (1952).

Jones, D. S. A simplifying technique in the solution of a class of diffraction problems. Quart. J. Math., Oxford Ser. (2) **3**, 189–196 (1952).

It is well known that a class of diffraction problems give rise to integral equations of the Wiener-Hopf type. The author shows that one may by-pass the integral equation by applying the transform in question directly to the partial differential equation. After a certain amount of manipulation, one is brought back to the same function-theoretic problem that one encounters in the original Wiener-Hopf work. Such views were expressed in part by M. Kantorovich and N. N. Lebedev [Akad. Sci. USSR J. Phys. **1**, 229–241 (1939)]. *A. E. Heins* (Pittsburgh, Pa.).

Povzner, A. Ya. On Cauchy's problem. Uspehi Matem. Nauk (N.S.) **7**, 5(51), 229–233 (1952). (Russian)

The solution of the equation $-\nabla^2u+\partial^2u/\partial t^2=f(x, t)$, where $x=(x_1, x_2, \cdots, x_n)$, $t\geqq\varphi(x)$, is found in terms of initial conditions on the hypersurface $t=\varphi(x)$, under certain restrictions of differentiability, etc. The solution makes use of a Fourier transform method and an expression for the fundamental solution of $\nabla^2u+\lambda^2u=0$. It is extended to give an existence theorem for solutions of $Lu+\partial^2u/\partial t^2=f(x, t)$ where L is an elliptic operator semibounded below.

J. L. B. Cooper (Cardiff).

***Hadamard, Jacques. Lectures on Cauchy's problem in linear partial differential equations.** Dover Publications, New York, 1953. iv+316 pp. Paperbound \$1.70. Clothbound \$3.50.

Photo-offset reprint of a book originally published by the Yale University Press [New Haven, 1923].

Ladyženskaya, O. On the convergence of Fourier series defining a solution of a mixed problem for hyperbolic equations. Doklady Akad. Nauk SSSR (N.S.) **85**, 481–484 (1952). (Russian)

The mixed initial-boundary value problem for the equation

$$\frac{\partial^2u}{\partial t^2}=\sum_{i,j=1}^n\frac{\partial}{\partial x_i}\left(a_{ij}(X)\frac{\partial u}{\partial x_j}\right)-a(X)u+f(X, t)\equiv Lu+f$$

is considered in a cylinder $Q=\Omega\times(0\leqq t\leqq l)$ where Ω is a bounded connected domain in $X=(x_1, x_2, \cdots, x_n)$-space. The idea of a generalized solution is introduced and a uniqueness theorem is stated for this generalized class of

solutions. By splitting off the time variable and considering the eigenfunctions of the elliptic operator L a formal solution can be obtained in terms of a Fourier series. Under very mild conditions on the coefficients and the given data it is stated that the corresponding Fourier series converges uniformly with respect to t to a function which is a generalized solution. Necessary and sufficient conditions for the termwise differentiability of the series are given.

M. H. Protter (Princeton, N. J.).

Hartman, Philip, and Wintner, Aurel. On hyperbolic partial differential equations. Amer. J. Math. 74, 834–864 (1952).

The authors show that the assumption of a Lipschitz condition with respect to z can be omitted in the existence theorems for the hyperbolic differential equation

$$(*)\qquad z_{xy}=f(x,y,z,p,q)\quad (p=z_x,\ q=z_y)$$

though not in the uniqueness theorems. On the other hand, an example is given in this paper showing that the Lipschitz condition with respect to (p,q) cannot be omitted and that there exist continuous functions f such that $(*)$ has no solution whatsoever. A function $z=z(x,y)$ is said to be of class C^* if $z(x,y)$ is of class C^1 and possesses a continuous mixed derivative $z_{xy}=z_{yx}$. Every C^1-solution of $(*)$ is a solution of class C^*. On the other hand, the solution $F(x)+G(y)$ shows that $z_{xy}=0$ can have solutions which are not of class C^1 if F, G possess first derivatives having discontinuities. If $f(x,y)$ is a continuous function in a vicinity of $(x,y)=(0,0)$ the second derivative Φ_{xx} of the function $\Phi(x,y)=\int_0^x\int_0^y f(\alpha,\beta)d\alpha d\beta$ will not in general exist, so that $\Phi(x,y)$ fails to be of class C^2 though it is of class C^*. It can be shown that if $F(x,y,z,p,q,r,s,t)$ is a function of class C^3, and if the partial differential equation

$$(**)\qquad F(x,y,z,p,q,r,s,t)=0\quad (r=z_{xx},\ s=z_{xy},\ t=z_{yy})$$

is of hyperbolic type, then initial data belonging to a z of class C^3 determine a unique C^3-solution of $(**)$. When $(**)$ is of the Monge-Ampère-type, it can be shown that if the coefficients are of class C^1, then initial data belonging to a z of class C^2 determine a unique C^2-solution of the type under consideration. In part II the results of part I are applied to two problems of differential geometry; first, the problem of embedding into 3-dimensional Euclidean space a positive-definite line element,

$$(***)\qquad ds^2=g_{11}du^2+2g_{12}dudv+g_{22}dv^2$$

of negative curvature; second, the question of transforming $(***)$ into the Tchebycheff form $ds^2=d\varphi^2+2\cos\varphi dxdy+dy^2$, $\varphi=\varphi(x,y)$. The authors obtain the following theorem: let $g_{ik}(u,v)$ be a binary, symmetric positive-definite matrix of class C^3 in a vicinity of $(u,v)=(0,0)$. Then there exists a vector function $X=(x(u,v),y(u,v),z(u,v))$ of class C^2 in a vicinity of $(u,v)=(0,0)$ with the property that $|dX|^2=g_{11}du^2+2g_{12}dudv+g_{22}dv^2$ holds. As observed by Bianchi, the possibility of transforming $(***)$, when $|g_{ik}|\neq 0$ into the normal form $ds^2=dx^2+2\gamma(x,y)dxdy+dy^2$ follows from an existence theorem of classical type if the functions g_{ik} have continuous partial derivatives of first order satisfying a uniform Lipschitz condition. The authors show that the imposition of a Lipschitz restriction is superfluous. In part III the authors adapt the method of part I to the quasi-linear hyperbolic system for the vector $z=\{z^1,z^2,\cdots,z^n\}$, $z_x=fz_y+g$ (x, y scalars, f, g vectors), and discuss especially the case that $f(x,y,z)$ be of class C^1 only.

M. Pini.

Pini, Bruno. Un problema di valori al contorno, generalizzato, per l'equazione a derivate parziali lineare parabolica del secondo ordine. Rivista Mat. Univ. Parma 3, 153–187 (1952).

For each fixed value of t $(0\leqq t<\delta)$ let $S(t)$ be the surface given by the equations: $x_i=x_i(\alpha,\beta;t)$ $(i=1,2)$, $y=\beta$, where α, β range over the rectangle $R\equiv(y_1\leqq\beta\leqq y_2;\ \alpha_1=\alpha=\alpha_2)$. For each fixed β the simple closed continuous curve $x_i=x_i(\alpha,\beta;t)$ lies interior to the curve $x_i=x_i(\alpha,\beta;t')$ if $t>t'$, and $\lim_{t\to 0}x_i(\alpha,\beta;t)=x_i(\alpha,\beta;0)$ uniformly in R. Let D be the domain limited by $S(0)$, $y=y_1$, and $y=y_2$, and consider the parabolic equation

$$(1)\qquad L(u)\equiv\sum_{i,i=1}^{2}a_{ij}\frac{\partial^2u}{\partial x_i\partial x_j}-\frac{\partial u}{\partial y}+au=f(x_1,x_2,y).$$

The a, $\partial a_{ij}/\partial y$, $\partial^2a_{ij}/\partial x_h\partial x_k$, satisfy a Lipschitz condition in x_1, x_2, y in a domain containing D and $\sum a_{ij}\lambda_i\lambda_j$ is a positive definite form. A function $u=u(x_1,x_2,y)$ which together with the $\partial u/\partial x_i$ are absolutely continuous in D, and which in D satisfy $L(u)=f$ almost everywhere will be termed a generalized solution of (1). Under certain continuity conditions on the surfaces $S(t)$ the following result is obtained. If f is a bounded and measurable function, and if $F(\alpha,\beta)\in\mathfrak{L}^r$ $(r>1)$ in R, then there exists one and only one generalized solution u of (1) such that $u=0$ for $y=y_1$, and

$$\lim_{t\to 0}\iint_R|u(x_1(\alpha,\beta;t),x_2(\alpha,\beta;t),\beta)-F(\alpha,\beta)|^r d\alpha d\beta=0.$$

A similar generalized boundary value problem for the equation $\partial^2u/\partial x^2+\partial u/\partial y+au=f$ was treated earlier by the author [Ann. Mat. Pura Appl. (4) 32, 179–204 (1951); these Rev. 13, 750].

F. G. Dressel (Durham, N. C.).

Blackman, Jerome. The inversion of solutions of the heat equation for the infinite rod. Duke Math. J. 19, 671–682 (1952).

Boundary value problems for the heat equation (1) $u_{xx}=u_t$ and for the behaviour of its classical solution

$$(2)\qquad u(x,t)=(4\pi t)^{-1/2}\int_{-\infty}^{\infty}\varphi(\tau)\exp\left[-(x-\tau)^2/4t\right]\varphi(\tau)d\tau$$

are discussed, involving, in particular, extensions to complex values $z=x+iy$ of the first variable.

If A is a closed region with the square $0\leqq x\leqq 1$, $0\leqq t<1$ in its interior, and $u(x,t)$ is a solution of (1) continuous in the interior of A, then, under certain conditions on $u(0,t)$ and $u(1,t)$, the solution (2) may be extended to all z inside the square in the complex z-plane which has as diagonal the segment $0\leqq x\leqq 1$, and $\lim_{t\to 1-}u(z,t)$ exists and is analytic in the square.

If $F(\xi)(\xi=\sigma+i\rho)$ is analytic for $a\leqq\sigma\leqq b$, and

$$F(\xi)\leqq g_1(\sigma)\exp\left[\rho^2g_2(\sigma)\right],$$

where g_1, g_2 are bounded in every finite subinterval of (a,b), and T positive and sufficiently small, then there is a $u(x,t)$ of form (2) such that $F(\xi)=\lim_{t\to T-}u(\xi,t)$, $a\leqq\sigma\leqq b$. Tychonoff [Mat. Sbornik 42, 199–216 (1935)] showed that if the integral (2) converges for x_0, t_0, it converges for all x and t with $0\leqq t<t_0$. Examples are constructed in which (2) converges for $t<t_0$ but not for $t=t_0$, while $u(x,t)$ can be extended beyond $t=t_0$ to give a solution of (1) in the region of extension. An application is given to inversion of a Weierstrass summation of a Fourier integral.

J. L. B. Cooper (Cardiff).

Paterson, Stewart. Propagation of a boundary of fusion. Proc. Glasgow Math. Assoc. 1, 42–47 (1952).

The problem considered is that of the propagation of a boundary of fusion in the two-phase case. A volume of material is divided into two regions 1 and 2, each of density ρ, by a moving surface S. On S a change of phase occurs at a definite temperature, zero. If θ_i, k_i, K_i $(i=1, 2)$ are the temperature, thermal conductivity, and diffusivity of phase i, then the following equations must hold:

$$(1)\qquad \frac{\partial\theta_i}{\partial t}=K_i\nabla^2\theta_i \quad (i=1, 2),$$

with the boundary conditions on S:

$$(2)\qquad \theta_1(x, y, z, t)=\theta_2(x, y, z, t)=0,$$

$$(3)\qquad k_1|\text{grad}\,\theta_1|-k_2|\text{grad}\,\theta_2| =\pm\frac{L\rho}{|\text{grad}\,\theta_1|}\frac{\partial\theta_1}{\partial t}=\pm\frac{L\rho}{|\text{grad}\,\theta_2|}\frac{\partial\theta_2}{\partial t}.$$

The condition for a solution is that θ_i separately satisfy

$$(4)\qquad \frac{\text{grad}^2\,\theta}{\partial\theta/\partial t}=\text{constant, when } \theta=0.$$

In the one-dimensional cases of linear, axially symmetric or spherically symmetric flow, equations (1) reduce to

$$(5)\qquad \frac{\partial\theta_i}{\partial t}=K_i\left[\frac{\partial^2\theta_i}{\partial r^2}+\frac{n}{r}\frac{\partial\theta_i}{\partial r}\right] \quad (i=1, 2),$$

with $n=0$, 1, 2 in the three cases, respectively. Further, condition (4) reduces to

$$(6)\qquad \left(\frac{\partial\theta}{\partial r}\right)^2\Big/\frac{\partial\theta}{\partial t}=\text{constant, at } \theta=0.$$

Using the known solutions for these three cases, the author exhibits solutions for the propagation of a boundary of fusion from a line source of heat in an infinite fusible solid and from a point source. An ice-water system is given as an example of the first problem and a temperature profile chart is constructed. It is pointed out that a boundary of vaporization should follow the boundary of fusion and an indication of the solution in this case is given. *C. G. Maple.*

Sestini, Giorgio. Esistenza ed unicità nel problema di Stefan relativo a campi dotati di simmetria. Rivista Mat. Univ. Parma 3, 103–113 (1952).

This continues the study begun in an earlier paper [same Rivista 3, 3–23 (1952); these Rev. 14, 381]. There the one-dimensional problem of the shifting boundary between two phases of a substance was investigated. Here cylindrical and spherical shells (with axially or centrally symmetric conditions) are treated. The equations for radial heat flow are special cases (for $m=2$ and 3 respectively) of

$$k\frac{\partial}{\partial r}\left[r^{m-1}\frac{\partial u}{\partial r}\right]-r^{m-1}\frac{\partial u}{\partial t}=0.$$

This general equation, with boundary conditions analogous to those in the former paper, is then studied. The region is that between $r=R_1$, where heat is supplied, and $r=R_2$, which is insulated $(R_2>R_1)$; $r=\alpha(t)$ is the boundary between solid and liquid.

The method of solution, as before, is that of successive approximations, with $u_{n-1}^{(1)}$ and $u_{n-1}^{(2)}$ furnishing α_n, which in turn determines $u_n^{(1)}$ and $u_n^{(2)}$. $\{\alpha_{2n}\}$ form a monotonically non-increasing sequence of limit α^{**}; $\{\alpha_{2n-1}\}$ a non-increasing one of limit α^*. Equality of these limits, and with it existence of a unique solution, are assured if t is so small that

$$H(t)\int_0^t H(\tau)d\tau<\rho_2{}^2k_1L^2.$$

(As in the earlier paper, ρ_2 is the outer density, k_1 the inner coefficient of thermal diffusivity, L the latent heat of fusion, $H(t)$ the rate of heat delivered at $r=R_1$.) *E. S. Allen.*

Sneddon, Ian N. Solutions of the diffusion equation for a medium generating heat. Proc. Glasgow Math. Assoc. 1, 21–27 (1952).

The heat equation in the form $\theta_t=k\nabla^2\theta+\phi(x, y, z, t)+\psi(t)\theta$, corresponding to the generation of heat within the medium at a rate that varies linearly with the temperature $\theta(x, y, z, t)$, is reduced to the form $u_t=k\nabla^2u+\chi(x, y, z, t)$ by a change of variables. When the functions ϕ and ψ are prescribed, the boundary-value problem for the temperatures θ in an infinite solid whose initial temperature is a given function $\theta_0(x, y, z)$ is solved with the aid of three-dimensional Fourier transforms. The formula for θ is simplified by using the resultant or convolution property of that transform. Special cases are noted, including the case of axially symmetric flow of heat which is also treated by means of Hankel transforms. Other Fourier transforms are used to derive the formula for the temperatures $\theta(x, y, z, t)$ in a semi-infinite solid $x\geqq 0$ when the face $x=0$ is insulated and when the temperature of that face is a prescribed function of y, z, and t. In these cases, again the initial temperature is arbitrary and the rate of generation of heat within the body is the above linear function of the temperature.

R. V. Churchill (Ann Arbor, Mich.).

Krüger, M. Zur Kombination thermischer und elektromagnetischer Felder im Falle der ebenen Platte. Ing.-Arch. 20, 234–246 (1952).

The control of temperatures in a plate in which heat is generated by a high-frequency alternating current is examined theoretically. Let $\theta(z, t)$ denote the temperatures within a plate of infinite extent whose faces are the planes $z=0$ and $z=d$. Heat transfer according to Newton's law of cooling takes place at those faces into media whose temperatures are prescribed functions of the time t. The initial temperature of the plate is a prescribed function of z. Let another function of z represent the rate of generation of heat within the plate; thus $\theta_t=a^2\theta_{zz}+f(z)$ when $0<z<d$ and $t>0$. This boundary-value problem is solved for $\theta(z, t)$ with the aid of the Laplace transformation. The author then shows how the function $f(z)$ or some of the prescribed functions, the initial temperature or the temperatures of the adjacent media, can be determined so that at a given instant $t=t_e$ the temperatures in the plate will be a prescribed function $g(z)$. In particular, the source function $f(z)$ is determined so that $g(z)=\theta_0$, a constant, when the initial temperature is zero and the faces are kept at the temperature θ_0. When the plate is a dielectric and the source function $f(z)$ arises from a quasi-stationary electromagnetic field, it is shown that, under certain restrictions, $f(z)=k_1e^{kz}$, where k_1 and k denote constants. In this case a uniform plate temperature θ_0 at $t=t_e$ can be obtained by adjusting initial and boundary conditions only when $k=0$. *R. V. Churchill.*

Yih, Chia-Shun. On a differential equation of atmospheric diffusion. Trans. Amer. Geophys. Union 33, 8–12 (1952).

En construisant une théorie de la diffusion atmosphérique O. G. Sutton [Proc. Roy. Soc. London. Ser. A. 146, 701–722

(1934)] a utilisé une hypothèse très simple $u=u_1(y/y_1)^n$ liant la vitesse moyenne u au niveau y avec sa valeur u_1 au niveau y_1, ce qui a permis de donner à l'équation de diffusion la forme

$$y^m \cdot \frac{\partial c}{\partial x}=K\frac{\partial}{\partial y}\left(y^n\frac{\partial c}{\partial y}\right)$$

avec $n=1-m$, $K=A_1 y_1^{2m-1}u_1^{-1}$, c=concentration de vapeur d'eau, x=coordonnée spatiale horizontal comptée à partir d'une ligne de sources disposée sur une surface régulière. L'application de l'analyse dimensionnelle permet de discuter et de résoudre cette équation. L'auteur se sert de la même équation en supposant m indépendant de n et en donnant à K la forme $K=A_1 y_1^{m-n}u_1^{-1}$. Il obtient de cette façon les solutions de l'équation dans trois cas: 1) diffusion à partir d'une ligne des sources disposée sur une surface régulière, 2) diffusion à partir d'une surface régulière, 3) évaporation continue sur une surface régulière. *V. A. Kostitsin.*

Yih, C. S. Similarity solution of a specialized diffusion equation. Trans. Amer. Geophys. Union 33, 356–360 (1952).

Après avoir étudié la diffusion atmosphérique dans le cas simple de deux dimensions [voir l'analyse ci-dessus], l'auteur considère le même problème en trois dimensions. Soient x, y, z trois coordonnées spatiales, dont x est horizontale dans la direction du vent, y est verticale, et z est horizontale perpendiculaire à la direction due vent. En supposant la vitesse et les coefficients de diffusion proportionnels à des certains puissances de y et en désignant par c la concentration de vapeur, on trouve l'équation différentielle

$$y^m\frac{\partial c}{\partial x}=D_1\frac{\partial}{\partial y}\left(y^n\frac{\partial c}{\partial y}\right)+D_2y^k\frac{\partial^2 c}{\partial z^2}.$$

L'application de l'analyse dimensionnelle permet de trouver une solution effective de cette équation, en supposant $m=k$, dans le cas d'une source ponctuelle. *V. A. Kostitsin.*

Ryaben'kil, V. S. On the application of the method of finite differences to the solution of Cauchy's problem. Doklady Akad. Nauk SSSR (N.S.) 86, 1071–1074 (1952). (Russian)

Consider the system of differential equations

$$(*)\quad \frac{\partial u_i}{\partial t}=\sum_{j=1}^{N}\sum_{\eta_1,\cdots,\eta_n}A_{ij}^{\eta_1,\eta_2,\cdots,\eta_n}(t)\frac{\partial^{\eta_1+\eta_2+\cdots+\eta_n}u_j}{\partial x_1^{\eta_1}\partial x_2^{\eta_2}\cdots\partial x_n^{\eta_n}}+f_i(t,x_1,\cdots,x_n),\quad i=1,2,\cdots,N,$$

with initial Cauchy data $u_i(0,x_1,\cdots,x_n)=\varphi_i(x_1,\cdots,x_n)$, $i=1,\cdots,N$, the coefficients $A_{ij}^{\eta_1,\eta_2,\cdots,\eta_n}(t)$ of (*) being defined for $0\leqq t\leqq T$ and the functions $f_i(t,x_1,\cdots,x_n)$ for $0\leqq t\leqq T$ and all real $x_1,\cdots,x_n$. The author introduces certain systems of difference equations, related to (*), and referred to as "difference schemes" for (*). Under certain conditions a difference scheme for system (*) is called "uniformly A-stable". The first theorem shows how the solution of (*) may be approximated by the solution of a uniformly A-stable difference scheme, provided, among other things, that the coefficients of (*) are sufficiently smooth. The next two theorems show how suitable uniformly A-stable difference schemes may be obtained when (*) is hyperbolic or p-parabolic, in the terminology of I. G. Petrowsky [Bull. Univ. d'Etat Moscou. Ser. Internat. Sect. A. 1, no. 7 (1938)]. *J. B. Diaz* (College Park, Md.).

Leray, Jean. Les solutions élémentaires d'une équation aux dérivées partielles, à coefficients constants. C. R. Acad. Sci. Paris 234, 1112–1115 (1952).

Let $a(\xi)=a(\xi_1,\cdots,\xi_l)$ $(l>2)$ be a polynomial of degree m. The open connected parts Δ_α of the set of real points ξ for which $a(\xi+i\eta)\neq 0$ when η is real are convex, and the function $d(\xi)=\sup_\eta|a^{-1}(\xi+i\eta)|$ is bounded on compact subsets of Δ_α. The equation $a(\partial/\partial x_1,\cdots)u(x)=v(x)$ has a unique solution such that

$$\|u(x)\exp(-x\cdot\xi)\|_2\leqq d(\xi)\|v(x)\exp(-x\cdot\xi)\|_2\quad(\xi\in\Delta_\alpha),$$

and the inverse Laplace transform of $a^{-1}(\xi+i\eta)$ is a distribution K_α, the elementary solution of a with respect to Δ_α, such that $u(x)=K_\alpha * v(x)$. If there is one Δ_α, say Δ_1, whose director cone Γ_1 has a non-empty interior (hyperbolic case), then there is another, say Δ_2, such that $\Gamma_2=-\Gamma_1$, and if, in addition, the set $W(a)$ of points $\xi+i\eta$ such that $a(\xi+i\eta)=0$ has no singularities, the elementary solution K_x with respect to Δ_1 vanishes outside the dual cone C_1 of Γ_1, while outside $-C_1$, $K_x=(2\pi i)^{1-l}f(\partial/\partial x_1,\cdots)\int_\Omega f^{-1}(\zeta)e^{x\zeta}\omega_a(\zeta,d\zeta)$; here Ω is the part of $W(a)$ whose real projection belongs to a vector ξ^* joining Δ_2 with Δ_1; $\omega_a(\zeta,d\zeta)$ is an exterior differential form such that $da(\zeta)\cdot\omega_a(\zeta,d\zeta)=d\zeta_1\cdots d\zeta_l$; the orientation of Ω is such that $i^{1-l}\sum\xi_k^*[\partial a(\zeta)/\partial\zeta_k]\cdot\omega_a(\zeta,d\zeta)>0$; f is a polynomial of degree $>l-m$ such that some neighbourhood of Ω is free of points of $W(f)$; the integral does not depend on the choice of ξ^* and f. When a is homogeneous, a similar formula for K_x holds, which can be further reduced and leads to invariant forms of formulas given by Herglotz [Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Phys. Kl. 78, 93–126, 287–318 (1926); 80, 69–114 (1928)] and Petrowsky [Mat. Sbornik N.S. 17(59), 289–370 (1945); these Rev. 8, 79] and to a new formulation of the Petrowsky condition for a stable lacuna. Proofs are given in a lecture series by the author, "Symbolic calculus with several variables, projections and boundary value problems for differential equations" [Inst. for Adv. Study, Princeton, N. J., 1951–52]. *L. Gårding* (Lund).

Tanzi Cattabianchi, Luigi. Una classe di equazioni alle derivate parziali generalizzanti l'equazione di Bessel, e risoluzione in un caso particolare notevole. Rivista Mat. Univ. Parma 3, 189–201 (1952).

The author presents an operational method for finding solutions of the partial differential equation

$$(x_1+\cdots+x_m)D^2z+2mnDz-\alpha^2(x_1+\cdots+x_n)z=0,$$

$$D=\frac{\partial}{\partial x_1}+\cdots+\frac{\partial}{\partial x_m}$$

in case n is a positive integer and α is a constant. *F. G. Dressel* (Durham, N. C.).

Heilbronn, Georges. Sur la construction des équations $s+f(x,y,z,p,q,r)=0$ qui possèdent un invariant du second ordre. C. R. Acad. Sci. Paris 235, 1090–1092 (1952).

In a preceding note with the same title [same C. R. 208, 1380–1382 (1939)] the author considered invariants of the second order of the equation $s+f=0$, relative to the explicit characteristics $y=\beta$, of the form $F(\beta)=A\cdot q_\beta+B$, and found that certain relationships must hold among the derivatives. In the present note, some of these results are simplified. Invariants relative to the implicit characteristics α, obtained from the equation $x_\beta=f_r$ are considered in special cases. For example, if $F(\alpha)=A(x,y,z,p,r)\cdot x_\alpha$, then the equation

must be linear: $s=ar+b$, and A must satisfy an equation which is linear in r. No proofs are given.
D. L. Bernstein (Rochester, N. Y.).

Difference Equations, Special Functional Equations

Hahn, Wolfgang. Über uneigentliche Lösungen linearer geometrischer Differenzengleichungen. Math. Ann. **125**, 67–81 (1952); Berichtigung, **125**, 324 (1953).

The author considers the q-difference equation (1) $\sum_{\nu=0}^{n} p_\nu(x) f(q^\nu x)=0$ ($|q|<1$); on letting

$$\theta f(x)=[f(qx)-f(x)]/x(q-1),$$

this takes the form (1′) $\sum_{\nu=0}^{n} P_\nu(x)\theta^\nu f(x)=0$. Solutions are found in the form of a series (2) $\sum_0^\infty A_i(1-ax)_{-i}$, where $(1-ax)_\lambda=\prod_0^\infty(1-aq^i x)(1-aq^{i+\lambda}x)^{-1}$. A complete set of n solutions can be found expressible by series of the form (2), convergent (and analytic) in certain ring-shaped regions of center at the origin; all other solutions are linear combinations, with q-periodic coefficients $\kappa(x)[=\kappa(qx)]$ of these n solutions. It is assumed that the coefficients in (1), (1′) are rational. Some of the results are as follows. Not every equation (1) has i.s.'s (improper solutions), that is, solutions of the form $g_H(x)=\sum_{-\infty}^{\infty} B_i(1-ax)_{H-i}$, H real. If i.s.'s exist, they have the following properties. The B_i satisfy a recurrence relation, characteristic for (1); the i.s.'s satisfy a certain initial equation and they are proper solutions of a certain q-difference equation of higher order; the i.s.'s terminated on the right are proper; those terminated on the left are essentially entire; when in (1′) one lets $q\to 1$, there results a differential equation, while the (suitably normed) i.s.'s, terminated on the left, go into solutions of this equation. A number of applications are made, including some to the Heine series. It might be of interest to establish a connection between the developments due to the author, and the general asymptotic theory of q-difference equations (near a singular point), due to the reviewer [Acta Math. 61, 1–38 (1933)]. *W. J. Trjitzinsky* (Urbana, Ill.).

Mišoň, Karel. Definition of the Bernoulli numbers. Sum of an arithmetic series without use of difference series. Časopis Pěst. Mat. **76**, 199–200 (1951). (Czech)

The author points out that the difference equation $\phi(p, x+1)-\phi(p, x)=x^p$, where p and x are integers and which has the particular solution $F(p, x)=\sum_{n=0}^{x-1} n^p$, may be solved as a polynomial in x in which the constant term is arbitrary. Since F and ϕ can differ by at most a periodic function of x, which may be taken as a constant in this case, it follows that F is a polynomial of degree $p+1$ in x whose coefficients are proportional to the Bernoulli numbers. In this way the author claims to avoid the notion of difference.
D. H. Lehmer (Los Angeles, Calif.).

Leont'ev, A. F. Differential-difference equations. Amer. Math. Soc. Translation no. **78**, 33 pp. (1952).

Translated from Mat. Sbornik N.S. 24(66), 347–374 (1949); these Rev. 11, 113.

Carlitz, L. Note on a paper of Bagchi and Chatterjee. Amer. Math. Monthly **59**, 683–684 (1952).

It is shown that if $f(u)$, not identically a constant, is meromorphic and satisfies the functional equation

$$f(u+v)-f(u-v)=2f'(u)f(v)/(1-k^2f^2(u)f^2(v)),$$

then $f(u)$ is the elliptic function λ^{-1} sn $(\lambda u, k^2/\lambda^4)$ (λ=arbitrary constant). *I. M. Sheffer* (State College, Pa.).

Functional Analysis

Kneser, Hellmuth. Konvexe Räume. Arch. Math. **3**, 198–206 (1952).

The author presents an axiomatic treatment of a convex space over a skew field ordered by its positive elements and shows that every such space can be imbedded in a vector space with centroids preserved. The minimal imbedding is described. An application of the notion of convex space to a proof of the fundamental theorem of game theory will be given in another paper. *H. W. Kuhn* (Bryn Mawr, Pa.).

Weissinger, Johannes. Zur Theorie und Anwendung des Iterationsverfahrens. Math. Nachr. **8**, 193–212 (1952).

Let Ω be a complete metric space with distance $|\xi, \eta|$ and let T map Ω into itself and satisfy $|T\xi, T\eta|\leqq|T|\cdot|\xi, \eta|$. The author proves the following fixed-point theorem: If $\sum_{\nu=1}^{\infty}|T^\nu|$ is finite, then, for $\xi_\nu=T^\nu\xi_0$, ξ_0 arbitrary, $\lim_{\nu\to\infty}\xi_\nu=\xi$ exists and $T\xi=\xi$. As a corollary an error estimate for $|\xi, \xi_\nu|$ is given. Application is made to systems of linear equations, integral equations of the second kind, a problem in potential theory, the implicit function theorem, and to systems of ordinary differential equations satisfying a Lipschitz condition. [The theorem and its proof are essentially the same as in a paper of Caccioppoli's [Atti Accad Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) 11, 794–799 (1930)]; an expository account of this theorem together with a large number of applications may be found in a paper of Nemyckiĭ [Uspehi Matem. Nauk 1, 141–174 (1936), especially pp. 144–152]; see also chapter 2 of Miranda, Problemi di esistenza in analisi funzionale, Tacchi, Pisa. 1950 [these Rev. 12, 265] and Hamilton, Duke Math. J. 13, 113–121 (1946) [these Rev. 7, 488].] *J. V. Wehausen.*

Gol'dman, M. A., and Kračkovskiĭ, S. N. On the null-elements of a linear operator in its Fredholm region. Doklady Akad. Nauk SSSR (N.S.) **86**, 15–17 (1952). (Russian)

Let A be a linear transformation in a Banach space R and let $T_\lambda=E-\lambda A$, E the identity operator. Let Φ_A be the Fredholm domain for A, i. e., the (open, non-empty) set of λ's in the complex plane for which $T_\lambda=U+V$, where U has an inverse and V is completely continuous [these are the λ's for which the Fredholm alternative holds; cf. Nikol'skiĭ, Izvestiya Akad. Nauk SSSR. Ser. Mat. 7, 147–166 (1943); these Rev. 5, 187]. Let

$$N(\lambda)=E[x\,|\,T_\lambda{}^n x=0,\ n=1, 2, \cdots],$$

the set of null elements in F. Riesz's terminology. (If A is completely continuous, $N(\lambda)$ is finite-dimensional.) Theorem 1: If $\lambda\in\Phi_A$, a necessary condition for $N(\lambda)$ to be finite-dimensional is that T_λ have the representation above with $UV=VU$. Several other theorems combine to give the following theorem: For a given component G of Φ_A the dimension of $N(\lambda)$ is either finite for all $\lambda\in G$, in which case the eigenvalues in G are isolated, or else is infinite for all $\lambda\in G$, in which case all $\lambda\in G$ are eigenvalues.
J. V. Wehausen (Providence, R. I.).

Atkinson, F. V. A spectral problem for completely continuous operators. Acta Math. Acad. Sci. Hungar. **3**, 53–60 (1952). (Russian summary)

Sz.-Nagy, Béla. On a spectral problem of Atkinson. Acta Math. Acad. Sci. Hungar. **3**, 61–66 (1952). (Russian summary)

These two papers each give a proof of the fact that if V_r, $r=1, \cdots, n$, are n completely continuous transforma-

tions on a linear normed complete space to the same space, then the characteristic values of the transformation $T(\lambda)=I+\sum_{r=1}^{n}\lambda^r V_r$ (the values of λ for which $T(\lambda)f=0$ has non-trivial solutions) are isolated in the λ-plane. Atkinson's proof depends on the fact that if $\lambda=\lambda_0$ is a characteristic value, then there exists a ρ such that for $0<|\lambda-\lambda_0|<\rho$ the (finite) dimension of the null spaces corresponding to $T(\lambda)$ remains constant in λ. Sz.-Nagy rests his proof on the fact that $(2\pi i)^{-1}\int_C(I-sT)^{-1}ds$, where C is a closed curve in the s-plane at all points of which the inverse of $I-sT$ exists, defines a projection for each such C. He proves the lemma: If $S_0, S_1, \cdots, S_n, \cdots$ are linear transformations such that S_0 is completely continuous and there esists an a such that $\|S_n\|\leqq a^n$, then, for any λ, there exists an ϵ sufficiently small such that the spectrum of $S_0+\sum_n\epsilon^n S_n$ in the ϵ-neighborhood of λ is determined by a matrix of finite order whose elements $c_{ik}(\epsilon)$ are analytic in ϵ. He can then show that the finite part of the derived set of the characteristic values of $T(\lambda)$ is both open and closed. *T. H. Hildebrandt.*

Dunford, Nelson. Spectral theory. II. Resolutions of the identity. Pacific J. Math. 2, 559–614 (1952).

[For part I see Trans. Amer. Math. Soc. 54, 185–217 (1943); these Rev. 5, 39.] This paper is the full exposition, with proofs, of the author's research which was reported on at the International Congress in 1950, under the title "The reduction problem in spectral theory" [Proc. Internat. Congress Math., Cambridge, Mass., 1950, vol. 2, pp. 115–122, Amer. Math. Soc., Providence, R. I., 1952; these Rev. 13, 359]. A good many, but not all, of the main results of the present paper were announced in the report to the Congress.

The following notation is used: X a complex Banach space; T a bounded linear operator mapping X into itself; $T(\xi)=(\xi-T)^{-1}$ when $\xi\in\rho(T)$ (the resolvent set); $\sigma(T)$ the spectrum of T; $x(\xi)$ the maximal analytic extension of $T(\xi)x$; $\sigma(x)$ the complement of the set on which $x(\xi)$ is defined; for any set σ in the complex plane $[\sigma]=\{x|\sigma(x)\subset\sigma\}$; B the class of Borel sets in the plane; σ' the complement of σ. A resolution of the identity is defined as an operator-valued function E_e defined for $e\in B$, such that $E_e+E_{e'}=I$, $E_{e_1e_2}=E_{e_1}E_{e_2}$, and x^*E_ex is countably additive on B for each $x\in X$, $x^*\in X^*$. Certain further expected properties of E_e are derived from this definition. If T and E_e always permute, and if for each $e\in B$ the closure of e contains the spectrum of T when T is restricted to E_eX, then E_e is called a resolution of the identity for T. Such a resolution of the identity for T is unique if it exists.

The paper falls into three parts. In part I the special assumptions are: (1) $\sigma(T)$ is nowhere dense (this insures that $x(\xi)$ is single-valued); (2) $[\sigma]$ is closed if σ is closed; and (3) there is a constant $K=K(T)$ such that $|x|\leqq K|x+y|$ whenever $\sigma(x)$ and $\sigma(y)$ are disjoint. Let S_1 be the class of all sets σ such that $[\sigma]+[\sigma']$ is everywhere dense in X. Then there is a unique projection-valued function E_σ defined on S_1 with the properties $E_\sigma+E_{\sigma'}=I$, $E_\sigma E_{\sigma'}=0$, $|E_\sigma|\leqq K$, $E_\sigma x=x$ if $x\in[\sigma]$, $E_\sigma x=0$ if $x\in[\sigma']$, $TE_\sigma=E_\sigma T$, and the closure of σ contains the spectrum of T when T is restricted to $E_\sigma X$. Next, the author arrives by stages at a certain subset S_3 of S_1 which is a Boolean algebra and on which $E_\sigma x$ is countably additive for each x. The following additional hypotheses will then insure the existence of a resolution of the identity for T: X is weakly complete, and to each $\epsilon>0$ and each complex λ there corresponds a $\sigma\in S_3$ such that λ is an interior point of σ and the diameter of σ is less than ϵ.

In part II the effort is made to determine conditions in more manageable form to insure that T will have a resolution of the identity. It is assumed: (4) that $\sigma(T)$ lies on a closed rectifiable Jordan curve Γ_0 which can be imbedded in a certain way in a family of such curves, and that the growth of $T(\xi)$ is restricted in a certain way near this curve Γ_0. The effect of this is to insure that assumptions (1) and (2) are fulfilled, and that, if ξ is a point of Γ_0, and $\mathfrak{M}_\xi^n$, $\mathfrak{N}_\xi^n$ are the null manifolds and ranges, respectively of $(T-\xi)^n$, then $\mathfrak{M}_\xi^n$ and the closure of $\mathfrak{N}_\xi^n$ do not change when n is greater than some index depending on ξ. It is then assumed: (5) that for each ξ in a set dense on Γ_0 the linear manifold spanned by $\mathfrak{M}_\xi^n$ and the closure of $\mathfrak{N}_\xi^n$ is dense in X. Assumptions (3), (4), (5) and weak completeness of X insure the existence of a resolution of the identity for T.

Part III deals with integral representations and with further discussion of conditions for the existence of a resolution of the identity for T. For the integral formulas see the cited review of the author's Congress report. Various special conditions are given which will insure that assumption (5) is fulfilled. If X is reflexive and assumption (4) is fulfilled, T will have a resolution of the identity if and only if: (6) there is a constant K such that, for any $x\in X$ and $x^*\in X^*$ and any component σ of the set of singularities of $x^*x(\xi)$ the residue (defined by an integral) of this function on σ is in absolute value less than $K|x|\cdot|x^*|$.

The paper concludes with a discussion of a strengthening of condition (4) so that the operator T will be of what is called finite type. In particular, if T is of type 1, it has a resolution of the identity, and $T=\int_{\sigma(T)}\lambda dE_\lambda$.

A. E. Taylor (Los Angeles, Calif.).

Riesz, Frédéric. Sur la représentation des opérations fonctionnelles linéaires par des intégrales de Stieltjes. Comm. Sém. Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.] Tome Supplémentaire, 181–185 (1952).

See Riesz, Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] 21, no. 16 (1952); these Rev. 13, 562.

Aronszajn, N. Theory of reproducing kernels. Trans. Amer. Math. Soc. 68, 337–404 (1950).

There is assumed a Hilbert space consisting of a class F of complex-valued functions on a general set E, with the hermitian positive definite inner product $[f_1, f_2]$. The function $K(x, y)$ on $E\times E$ is a reproducing kernel (r.k.) if for every f of F: $f(y)=[f(x), K(x, y)]_x$, notions defined by the author in a previous paper [Proc. Cambridge Philos. Soc. 39, 133–153 (1943); these Rev. 5, 38]. If a reproducing kernel exists, then it is unique and a positive (hermitian) function (i.e., $K(x_i, x_j)$, $i, j=1, \cdots, n$, is a positive hermitian matrix for all $x_1, \cdots, x_n$). E. H. Moore [Mem. Amer. Philos. Soc. 1, pt. 2 (1939), Chap. V] has shown that any positive hermitian function K on $E\times E$ gives rise to a complete Hilbert space of functions on E with K as reproducing kernel. Conversely if $\mathfrak{H}$ is an abstract Hilbert space with inner product $[x, y]$ and one sets $E=\mathfrak{H}$, and $K(x, y)=[x, y]$, then the Moore process gives a complete Hilbert space $\mathfrak{H}'$ of functions on $\mathfrak{H}$, having $[x, y]$ as reproducing kernel, elements y of $\mathfrak{H}$ corresponding isometrically to the (linear) functions $y(x)=[x, y]$. A necessary and sufficient condition that F have a reproducing kernel is that $f(y)$ for fixed y be a linear continuous form on the Hilbert space F. This paper considers properties of reproducing kernels, assuming E, F and the r.k. K given. If F is finite-dimensional with $w_i(x)$, $i=1, \cdots, n$, a basis for F, then the r.k. for F is

$\sum \beta_{ij} w_i(x) \bar{w}_j(y)$, where β_{ij} is the inverse of the Gramian $\alpha_{ij} = [w_i, w_j]$. If K is a r.k. for F, then the class F_1 of restrictions of F to $E_1 \subset E$, has the restriction of K on $E_1 \times E_1$ as r.k. with $\|f_1\|_1 = \min \|f\|$, for all f of F equal to f_1 on E_1. If F_1 and F_2 are on E with r.k. K_1 and K_2 respectively, then the class of functions F of the form $f_1 + f_2$ has $K_1 + K_2$ as r.k., with $f = \min \|f_1\|_1 + \|f_2\|_2$ for all f_1, f_2 such that $f_1 + f_2 = f$. Similarly $K_1 \cdot K_2$ is the r.k. for the class F of all functions of the direct product $F' = F_1 \times F_2$ restricted to the diagonal set of $E \times E$, $K_1(x, y) \cdot K_2(x', y')$ being the r.k. for F'. The positive functions $K(x, y)$ form an ordered system with $K_1 \ll K_2$ if $K_2 - K_1$ is a positive function. If K is the r.k. for F and F_1 is a subclass of F with a norm such that $\|f\|_1 \leqq \|f\|$ for all f of F in F_1, then F_1 possesses an r.k. K_1 with $K_1 \ll K$. An increasing sequence of sets $E_1 \subset E_2 \subset \cdots$ with $E = \sum E_n$ gives rise to a decreasing sequence of classes $F_1 \supset F_2 \supset \cdots$ and a decreasing sequence of kernels $K_1 \ll K_2 \ll \cdots$ provided for every f_n of F_n and $m \leqq n$, the restriction of f_n to E_m belongs to F_m. Then the K_n converge to a kernel K_0 whose corresponding function class F_0 consists of functions whose restrictions to E_n belong to F_n and $\|f\|_n$ is bounded in n. For a decreasing sequence of sets $E_1 \supset E_2 \supset \cdots$ with $E = \prod E_n$ one determines the set E_0 as the set of y for which $\lim_n K_n(y, y) < \infty$, with F_0 the class of restrictions of f_n of F_n to E_0. If the completion of F_0 to F_0^* as a Hilbert space of functions on E_0 is possible, then the restrictions $K_{n0}(x, y)$ of K_n to $E_0 \times E_0$ converge to a function $K_0(x, y)$ which is the r.k. for F_0^*. Any resolution of the identity in a Hilbert space with an r.k. results in corresponding resolutions of the r.k. The bounded linear transformation L on a space F with an r.k. K induces a function $L^*K(x, y) = \Lambda(x, y)$ such that $Lf = [f, \Lambda(x, y)]$ whose properties reflect the properties of L. Finally there is developed in a space F with an r.k. a formula for the reproducing kernels of $F_1 + F_2$ in terms of the r.k. of the closed subspaces F_1 and F_2 of F. Since the r.k. of a subspace is determined by projections on this space, the formula in question is based on the equation

$$P = P_0 + \sum_{k=1}^{\infty} [P_1(P_2P_1)^{k-1} + P_2(P_1P_2)^{k-1} - (P_1P_2)^k - (P_2P_1)^k]$$

where P is the projection on $F_1 + F_2$, P_0 on $F_1 \cdot F_2$, P_1 on F_1 and P_2 on F_2.

The second part of the paper discusses examples of reproducing kernels, viz., (a) the Bergman kernels [Sur les fonctions orthogonales de plusieurs variables complexes . . . , Interscience, New York, 1941; these Rev. **2**, 359] determined by the space of all functions analytic on a domain D with $\|f\| = \int_D |f|^2 dxdy$, (b) the harmonic kernels determined by the space of all functions harmonic on a domain D with the same metric as (a), and (c) the Szegö kernels [Math. Z. **9**, 218–270 (1921); and Garabedian, Trans. Amer. Math. Soc. **67**, 1–35 (1949); these Rev. **11**, 340] determined by all functions analytic interior to a domain bounded by an analytic curve with $\|f\| = \int_C |f|^2 ds$, where C is the boundary of D. Among other things, the form of the harmonic kernel for the ellipse and the rectangular infinite strip are determined. Also there is an expression giving the relation between the Bergman kernel and the harmonic kernel for a given region based on the fact that any harmonic function can be expressed as the sum of an analytic function and the conjugate of an analytic function $h(z) = \varphi_1(z) + \overline{\varphi_2(z)}$, the representation being unique up to an additive constant.

T. H. Hildebrandt

Krein, M. G. Hermitian positive kernels on homogeneous spaces. I. Ukrain. Mat. Žurnal **1**, no. 4, 64–98 (1949). (Russian)

Part II was obtained first and has already been reviewed [same Žurnal **2**, no. 1, 10–59 (1950); these Rev. **12**, 719]. In the first of four sections the author discusses a number of properties of the set P_Q of continuous positive definite functions on $Q \times Q$ where Q is a topological space. The Hilbert space H_F associated with an $F \in P_Q$ [cf. the paper reviewed above] is introduced and the relationship between orthonormal systems in H_F and bilinear expansions of F is developed. Special attention is paid to the compact case and the case in which Q is the real line and F is differentiable. The section concludes with a study of the analytic functions in P_Q when Q is an open subset of the complex plane. For example, if $F \in P_Q$ and is analytic in an open set containing the diagonal of $Q \times Q$, then it is analytic throughout $Q \times Q$ and if F is analytic and in P_{Q_1} where $Q_1 \subset Q$, then the analytic continuation of F from $Q_1 \times Q_1$ to $Q \times Q$ is in P_Q.

Section two deals with the ring R_Q of all finite linear combinations with complex coefficients of members of P_Q. It can be normed so as to be a Banach algebra, and when Q is compact, the maximal ideals correspond one-to-one to the points of Q and the algebra is regular in the sense of Šilov.

In section three a transitive topological group of homeomorphisms of Q is postulated and a study is made of the set P_{QG} of members of P_Q which are invariant under G. Each member of P_{QG} is shown to define a unitary representation of G [cf. S. Ito, Proc. Japan Acad. **26**, no. 1, 17–28 (1950); these Rev. **12**, 242]. The section concludes with a characterization of the orthonormal systems associated with the bilinear expansions of members of P_{QG}.

The final section is concerned with "zonal functions", that is, with the extreme points of the convex set Π of all members Z of P_{QG} such that $Z(p, p) = 1$. It is shown that these are just those members of Π for which the corresponding representations of G are irreducible [cf. S. Ito, loc. cit.]. In the last half of the section the problem of decomposing a general member of P_{QG} as an integral of zonal parts is considered in four special cases; namely, those in which Q is Euclidean n-space, real Hilbert space, Lobachevskian n-space, and infinite-dimensional Lobachevskian space and G is the group of all isometries. Very explicit results are obtained.

G. W. Mackey (Cambridge, Mass.).

Fell, J. M. G., and Kelley, J. L. An algebra of unbounded operators. Proc. Nat. Acad. Sci. U. S. A. **38**, 592–598 (1952).

This paper is an extension of investigations by M. H. Stone [same Proc. **26**, 280–283 (1940), Canadian J. Math. **1**, 176–186 (1949); these Rev. **1**, 338; **10**, 546]. Let $\mathfrak{A}$ be a commutative algebra of bounded operators on a Hilbert space H. It is assumed that $\mathfrak{A}$ contains the identity operator 1, is self-adjoint, and closed in the strong topology for operators. It is well-known [Gelfand and Neumark, Mat. Sbornik N.S. **12**(**54**), 197–213 (1943); these Rev. **5**, 147] that $\mathfrak{A}$ is isometrically *-isomorphic to the algebra $\mathfrak{C}$ of all complex continuous functions on a compact Hausdorff space X. As a consequence of the strong closure of $\mathfrak{A}$, the space X is extremally disconnected in the sense that the closure of every open set is open and closed. Also, for any Borel set $S \subset X$ there exists a unique open and closed set S' such that the symmetric difference of S and S' is of the first category. Denote by T_f the operator in $\mathfrak{A}$ corresponding to f in $\mathfrak{C}$.

Next let $\bar{\mathfrak{C}}$ be the set of all continuous functions on X to the complex sphere which take the value ∞ only on a set of the first category. Under obvious definitions, $\bar{\mathfrak{C}}$ is an algebra isomorphic to the algebra of all complex Borel functions on X modulo the ideal of functions vanishing outside sets of the first category. The fundamental problem considered here is that of extending the mapping $f\to T_f$ to all of $\bar{\mathfrak{C}}$. Such an extension exists in the following sense. For each $f \in \bar{\mathfrak{C}}$ there exists an operator $\bar{T}_f$ on H with the properties: $\bar{T}_f$ is a densely defined closed operator which is unbounded except for $f \in \mathfrak{C}$ when $\bar{T}_f = T_f$; $\bar{T}_f^* = \bar{T}_{\bar{f}}$; the mapping $f\to\bar{T}_f$ provides an algebraic isomorphism of $\bar{\mathfrak{C}}$ with an algebra $\bar{\mathfrak{A}}$ of (unbounded) operators on H in the sense that the graphs of $\bar{T}_{fg}$ and $\bar{T}_{f+g}$ are respectively the closures of the graphs of $\bar{T}_f\bar{T}_g$ and $\bar{T}_f+\bar{T}_g$. The following form of the spectral theorem is obtained. For each Borel set S let $\mu(S) = T_e$ where e is the characteristic function of S'. Then μ is a projection-valued measure and, under natural definitions of the integrals, one has $T_f = \int f d\mu$, for $f \in \mathfrak{C}$, and $\bar{T}_f x = \int f d\mu^x$, for $f \in \bar{\mathfrak{C}}$ where μ^x is the vector-valued measure defined by $\mu^x(S) = \mu(S)x$. *C. E. Rickart* (New Haven, Conn.).

Kelley, J. L. Commutative operator algebras. Proc. Nat. Acad. Sci. U. S. A. **38**, 598–605 (1952).

Consider an algebra $\mathfrak{A}$ satisfying the conditions of the paper reviewed above. (We carry over the notations in that paper.) The author investigates the structure of $\mathfrak{A}$, the main result being a multiplicity theorem. For $x \in H$, define the "characteristic measure" μ_x for a Borel set S by $\mu_x(S) = (\mu(S)x, x)$. The characteristic measures are completely determined by X and its topology. A subset J of the unit sphere in H is called an "$\mathfrak{A}$-base" for a projection E if it satisfies the conditions: (a) For each $x \in J$, $x \neq 0$ and E is the infimum of all those projections in $\mathfrak{A}$ whose ranges contain x. (b) If x, y are distinct elements of J, then $\mathfrak{A}x \perp \mathfrak{A}y$. (c) J is maximal with respect to (a) and (b). A projection E is called "primitive" if there is an $\mathfrak{A}$-base J for E such that the sets $\mathfrak{A}x$ for $x \in J$ span the range of E. If also $\mu_x = \mu_y$ for every $x, y \in J$, then J is called a "proper $\mathfrak{A}$-base". Two proper $\mathfrak{A}$-bases for E must have the same cardinal number. Next a "multiplicity function" φ on X to the cardinal numbers is constructed such that, if U is an open and closed set with characteristic function e and if the projection T_e has a proper $\mathfrak{A}$-base J, then $\varphi(p)$ is equal to the cardinal number of J for every $p \in U$. The space X and the function φ are unitary invariants of $\mathfrak{A}$ in an obvious sense. The algebra $\mathfrak{A}$ can, of course, be reconstructed from X and φ. This treatment of multiplicity is closely related to one given by I. E. Segal [Mem. Amer. Math. Soc., no. **9** (1951); these Rev. **13**, 472] and less directly to one by Halmos [Introduction to Hilbert space and the theory of spectral multiplicity, Chelsea, New York, 1951; these Rev. **13**, 563]. *C. E. Rickart* (New Haven, Conn.).

Newburgh, J. D. The variation of spectra. Duke Math. J. **18**, 165–176 (1951).

The author studies the continuity properties of spectra of elements in a Banach algebra A. It is shown that if $x, x_i \in A$ and $x_i \to x$, then $\overline{\lim}\, s(x_i) \subset s(x)$. Here $s(y)$ designates the spectrum of y and $\overline{\lim}$ is defined in the sense of Hausdorff. Next, let s_0 be a non-empty spectral set of x and let $x_i \to x$. Then $\overline{\lim}\, s(x_i) \cap s_0$ is not empty. It follows that if x is a completely continuous operator, then the mapping s: $A \to S$ which carries x into $s(x)$ is continuous at x. A set $\Phi \subset A$ has property ϕ means that if $x \in \Phi$, then for all λ, $x + \lambda 1 \in \Phi$ and there exists a number $K > 0$ such that for each $x \in \Phi$, if x^{-1} exists, $|s(x^{-1})| \geqq K\|x^{-1}\|$. Let $x_i \to x$ where $x_i \in \Phi$ and Φ has property ϕ. Then $\lim s(x_i) = s(x)$. A study is also made of algebras of linear operators, and of closed operators. Finally, it is shown that in an algebra A without radical and in which the spectral radius of $x \in A$ is a continuous function of x, then any norm on A defining a new algebra A_1 is equivalent to the norm of A. It is also shown in a commutative algebra, $x_i \to x$ implies $\lim s(x_i) = s(x)$. Thus the theorem on norms constitutes an extension of a known result. *E. R. Lorch* (New York, N. Y.).

Kadison, Richard V. A generalized Schwarz inequality and algebraic invariants for operator algebras. Ann. of Math. (2) **56**, 494–503 (1952).

Let $\mathfrak{A}$ be a C^*-algebra (uniformly closed, self-adjoint operator algebra) and φ a linear order-preserving map of $\mathfrak{A}$ into the algebra of all bounded operators on some Hilbert space such that $\|\varphi\| \leqq 1$. The generalized Schwarz inequality proved here states that $\varphi(A^2) \geqq \varphi(A)^2$ for every self-adjoint operator in $\mathfrak{A}$. The author uses this inequality to prove the following theorem. Let $\mathfrak{A}_1$ be a second C^*-algebra and ρ an isometry of the Jordan algebra of self-adjoint elements in $\mathfrak{A}$ onto the Jordan algebra of self-adjoint elements of $\mathfrak{A}_1$. Then, when ρ is extended linearly to all of $\mathfrak{A}$, it has the form $\rho = U\varphi$ where U is a self-adjoint unitary operator in the center of $\mathfrak{A}_1$ (viz., $\rho(I)$) and φ is a C^*-isomorphism of $\mathfrak{A}$ onto $\mathfrak{A}_1$ (i.e., $\varphi(A^*) = \varphi(A)^*$ and $\varphi(A^2) = \varphi(A)^2$ [Kadison, Ann. of Math. (2) **54**, 325–338 (1951); these Rev. **13**, 256]). Next let $\mathcal{P}$ be the w^* closure of the set of extreme points of positive linear functionals on $\mathfrak{A}$ which take the value 1 on the identity operator I. The set $\mathcal{P}$ is called the pure state space of $\mathfrak{A}$. For $A \in \mathfrak{A}$ associate the function $A(\cdot)$ on $\mathcal{P}$ defined for $f \in \mathcal{P}$ by $A(f) = f(A)$. Then $A \leftrightarrow A(\cdot)$ is a linear isomorphism of $\mathfrak{A}$ with a subspace $\mathfrak{L}$ of $C(\mathcal{P})$. A corollary of the above theorem is that, if $\mathfrak{A}_1, \mathfrak{A}_2$ are C^*-algebras with pure state spaces $\mathcal{P}_1, \mathcal{P}_2$ and function representations $\mathfrak{L}_1, \mathfrak{L}_2$, respectively, and if $\mathcal{P}_1, \mathcal{P}_2$ are homeomorphic under a map which takes $\mathfrak{L}_1$ into $\mathfrak{L}_2$, then $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are C^*-isomorphic under a map which induces the given homeomorphism. Another corollary is that the function representation $\mathfrak{L}$ of $\mathfrak{A}$ on $\mathcal{P}$ is closed under pointwise multiplication if and only if $\mathfrak{A}$ is commutative. The following theorem is also proved. If φ is a linear map of $\mathfrak{A}_1$ into $\mathfrak{A}_2$ which sends I into I and $\varphi(|A|) = |\varphi(A)|$ for self-adjoint A, then φ is a C^*-homomorphism. The paper closes with a number of examples. *C. E. Rickart* (New Haven, Conn.).

Dixmier, J. Applications ♮ dans les anneaux d'opérateurs. Compositio Math. **10**, 1–55 (1952).

This paper gives detailed proofs for results previously announced [C. R. Acad. Sci. Paris **230**, 607–608 (1950); these Rev. **11**, 524]. The author extends the center-valued trace on a "finite" operator ring to a similar trace on general operator ring by procedures resembling those used by Murray and von Neumann for the treatment of factors. The main result is essentially that if the identity is the least upper bound of the finite projections (i.e., if the ring has no part of type III), then there exists a strongly dense ideal in the ring on which a faithful trace is defined, which preserves least upper bounds of monotone directed families of operators. This trace is essentially unique; any other such trace is proportional to it by an operator which is η the center. Numerical traces which preserve least upper bounds as before are shown to be all derivable from the center-valued trace via a functional on the center. Preservation of

least upper bounds here may be replaced by complete additivity on projections. *I. E. Segal* (Chicago, Ill.).

Michael, Ernest A. Locally multiplicatively-convex topological algebras. Mem. Amer. Math. Soc., no. 11, 79 pp. (1952). $1.40.

L'auteur se propose d'étendre la théorie des algèbres de Banach, due à Gelfand, à une classe plus vaste d'algèbres topologiques, qu'il appelle multiplicativement convexes (ou m-convexes): une telle algèbre A a une topologie d'espace localement convexe, pour laquelle il existe un système fondamental de voisinages U de 0, convexes, symétriques et tels que $UU \subset U$; cela entraîne la continuité de la somme et du produit dans A, mais il y a des algèbres topologiques localement convexes (même métrisables et complètes) qui ne sont pas m-convexes; toutefois, de nombreuses algèbres de fonctions sont m-convexes, par exemple, l'algèbre $\mathfrak{D}$ des fonctions indéfiniment dérivables à support compact, avec la topologie d'espace (LF), ou l'algèbre des fonctions holomorphes dans un domaine, avec la topologie de la convergence compacte. Le principe de la méthode suivie consiste à se ramener à la théorie de Gelfand: (U_i) étant un système fondamental de voisinages de 0, convexes symétriques et tels que $U_iU_i \subset U_i$ le plus grand sous-espace vectoriel N_i de A contenu dans U_i est un idéal (bilatère) fermé, et l'image canonique de U_i dans A/N_i définit sur cette algèbre quotient une structure d'algèbre normée; on considère alors A comme plongée dans le produit des algèbres normées A/N_i, et il s'agit de voir dans quelles conditions les propriétés de ces dernières peuvent se transférer à A. Le plus souvent, A est supposée en outre complète, et dans ce cas, l'auteur prouve que A est une limite projective d'algèbres de Banach A_i (savoir les complétions des A/N_i); il en résulte que pour que x soit quasi-régulier dans A, il faut et il suffit que sa projection dans A_i le soit pour tout i. Cela permet d'utiliser la théorie de Gelfand pour étudier le radical, les caractères, le spectre, et les idéaux maximaux de A; les résultats les plus complets sont naturellement obtenus lorsque A est en outre commutative, ou une *-algèbre; si en outre A est métrisable (et complète), elle a des propriétés qui se rapprochent encore davantage de celles des algèbres de Banach. Nous renvoyons au mémoire lui-même pour une description plus détaillée des résultats, qu'il serait trop long de donner ici; signalons en particulier l'étude de la continuité des caractères (§12), celle des éléments à spectre borné (§13), et celle des algèbres (LF) (§15). Plusieurs appendices, consacrés à des résultats auxiliaires, terminent le mémoire; l'appendice B est particulièrement intéressant, donnant des conditions pour qu'une sous-algèbre A d'une algèbre topologique B, dense dans B, soit telle que pour tout idéal à gauche fermé, régulier et maximal J de B, $J \cap A$ soit régulier maximal dans A.

J. Dieudonné (Ann Arbor, Mich.).

Arens, Richard. A generalization of normed rings. Pacific J. Math. 2, 455–471 (1952).

Le sujet de ce mémoire est le même que celui du travail de E. A. Michael résumé ci-dessus; l'auteur signale qu'il en avait annoncé les principaux résultats en 1946 [Bull. Amer. Math. Soc. 53, 69 (1947)], et, contrairement au mémoire de Michael, il ne cherche pas à étudier la question dans tous ses détails. Les méthodes sont substantiellement les mêmes; lorsque A est complète, l'auteur remarque que les A/N_i (notations du précédent rapport) ne le sont pas nécessairement; il améliore le résultat de Michael en prouvant que si la projection x_i de x dans chacun des A_i admet un quasi-inverse à droite unique, x admet un quasi-inverse unique dans A; lorsque, dans ce dernier énoncé, on supprime la condition d'unicité pour le quasi-inverse de x_i, on peut cependant encore dans certains cas affirmer l'existence d'un quasi-inverse de x. La condition que l'auteur introduit à cet effet (existence d'une "partition localement finie de l'unité" dans A) lui permet aussi de donner une caractérisation de l'algèbre $C(T)$ des fonctions complexes continues dans un espace localement compact et paracompact, $C(T)$ étant munie de la topologie de la convergence compacte; cette caractérisation généralise le théorème connu de Gelfand-Neumark. *J. Dieudonné* (Ann Arbor, Mich.).

Pták, Vlastimil. Partially ordered linear spaces. Časopis Pěst. Mat. 76, 283–290 (1951). (Czech)

Résumé of basic definitions employed and results obtained in the recent treatise of Kantorovič, Vulih, and Pinsker [Functional analysis in partially ordered spaces, Moscow-Leningrad, 1950; these Rev. 12, 340].

E. Hewitt (Seattle, Wash.).

Yamamuro, Sadayuki. On linear modulars. Proc. Japan Acad. 27, 623–624 (1951).

H. Nakano [Modulared semi-ordered linear spaces, Maruzen, Tokyo, 1950; these Rev. 12, 420] has studied partially ordered linear spaces in which a modular function $m(a)$ is defined. This modular generalizes the function $\sum_i x_i^p$, that is, the pth power of the customary norm, in l_p space. Nakano derived from the abstract modular, two norm functions $\|a\|$ and $|||a|||$ and showed that they coincide if the modular is linear or singular. Yamamuro shows that the two norms coincide only in these cases. As a comment on the general Nakano theory of modulars, the reviewer feels that the special case of l_p space could well be given greater emphasis; here the norm $|||a|||$ coincides with the customary norm and the ratio $|||a|||/\|a\|$ is a constant (depending on p). *I. Halperin* (Kingston, Ont.).

Calculus of Variations

★Èl'sgol'c, L. È. Variacionnoe isčislenie. [The calculus of variations.] Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1952. 167 pp. 5.30 rubles.

An introductory text for engineers, short and of high quality. Topics discussed include functionals depending on higher order derivatives, functionals with more than one independent variable, variable endpoint problems, extremals with corners, one-sided variation problems (extremal constrained to lie in a closed region), conditioned extrema, isoperimetric problems, Euler's finite difference method, the method of Ritz, and a method of Kantorovič. Everything is copiously illustrated with examples, and there are very many exercises. *J. M. Danskin.*

Bononcini, Vittorio E. Sugli integrali regolari del calcolo delle variazioni per superficie in forma parametrica. Rivista Mat. Univ. Parma 3, 131–151 (1952).

Let $F(x, p) \equiv F(x_1, x_2, x_3, p_1, p_2, p_3)$ be the usual integrand for parametric surface integrals, continuous in all 6 variables for x in a closed set A and p arbitrary, and positively homogeneous in the p's. Let $\mathfrak{s}(S)$ be the associated Weierstrass-Cesari integral. Cesari proved [Ann. Scuola Norm. Super. Pisa (2) 13, 77–117 (1948); these Rev. 9, 505] that if $S_n \to S$

and $L(S_n)\to L(S)$, then $\mathscr{s}(S_n)\to\mathscr{s}(S)$. The present result is a partial converse. Assume that the integral $\mathscr{s}(S)$ is positively regular and positive definite. Assume that the first and second partials of F with respect to the p's exist and are continuous, for $x \in A$ and $p\neq 0$. Let A_{ij} be the cofactor of $F_{p_ip_j}$ in the determinant $|F_{p_ip_j}|$. Assume that (*) $A_{ii}>0$ whenever $p_i>0$, $i=1, 2, 3$. Then if $S_n\to S$ and $\mathscr{s}(S_n)\to\mathscr{s}(S)$, $L(S_n)\to L(S)$. The author here imposes the special condition (*) on the integrand. A related result of Glebskiĭ [Mat. Sbornik N.S. **30**(72), 529–542 (1952); these Rev. **13**, 925] imposes no special conditions on the integrand, but imposes a condition on the representations of the S_n, viz. that they have Dirichlet representations with the Dirichlet integrals uniformly bounded. *J. M. Danskin.*

Viola, Tullio. Su una classe di problemi non regolari di calcolo delle variazioni, attinenti all'equazione $\Delta_2{}^n u=0$. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **12**, 511–518 (1952).

Let M: $(x_1, \cdots, x_r)$ be a generic point in r-dimensional Euclidean space. Put $\Delta_2{}^0=1$, $\Delta_2{}^1=\Delta_2=\sum_{i=1}^r\partial^2/\partial x_i{}^2$, $\Delta_2{}^n=\Delta_2(\Delta_2{}^{n-1})$, $n=1, 2, \cdots$. The n-hyperharmonic equation is (a) $\Delta_2{}^n u=0$. By using Green's theorem for hyperharmonic functions [see e.g. Nicolesco, Les fonctions polyharmoniques, Hermann, Paris, 1936] it is easy to see that a solution of (a) is formally the Euler equation of a minimum problem. (In the case $n=1$ this reduces to the classical representation of ordinary harmonic functions as Euler equations for the Dirichlet problem.) For $n=2p$ the appropriate minimum problem is given by (b) $I(u)=\int_A(\Delta_2{}^p u)^2 d\tau$. For $n=2p+1$ we take (c) $I(u)=\int_A(\text{grad } \Delta_2{}^p u)^2 d\tau$. The region of integration A is assumed here to be bounded, open, and connected. The frontier $\mathfrak{F}A$ is piecewise of class C^2, and regular in a sense which assures that functions defined and summable on the interior of A and having continuous first partial derivatives there take on limiting values on $\mathfrak{F}A$ in a way independent, almost everywhere, of the direction of approach.

For the case $n=2$, Fichera [same Rend. (8) **11**, 34–39 (1951); these Rev. **13**, 758] exhibited a solution of the minimum problem for (b) among the class of functions of class C^2 in A, with first and second partial derivatives of class L^2 in A, and taking on, together with their normal derivatives, prescribed boundary values. He proved that this solution satisfies the corresponding Euler equation $\Delta_2^2 u=0$.

The present paper announces a solution of the minimum problem for general n. For $n=2p$, a sufficiently smooth function u is admitted if $u, \Delta_2 u, \cdots, \Delta_2{}^{p-1}u$ and their normal derivatives take on prescribed values on $\mathfrak{F}A$. For $n=2p+1$, we require $u, \Delta_2 u, \cdots, \Delta_2{}^p u$, and the normal derivatives of $u, \Delta_2 u, \cdots, \Delta_2{}^{p-1}u$ to take on prescribed values on $\mathfrak{F}A$. The author gives explicit formulas for the solution in both the even and odd cases. His solutions are extremals, that is, solutions of the Euler equation $\Delta_2{}^n u=0$. Only a sketch of the proof is given, details being promised in a later paper.
J. M. Danskin (Washington, D. C.).

Stampacchia, Guido. Problemi al contorno per equazioni di tipo ellittico a derivate parziali e questioni di calcolo delle variazioni connesse. Ann. Mat. Pura Appl. (4) **33**, 211–238 (1952).

A function $f(x)$ defined in a bounded region R of E^n is μ_r-quasi continuous there if for any $\epsilon>0$ there is a set I such that f is continuous in $R-I$, and the measure of the projection of I on any r-dimensional coordinate subspace is less than ϵ. In an analogous way we define the concept of μ_r-quasi uniform convergence of sequences. A collection $\{f\}$ of functions is equi-μ_r-quasi bounded in R if for any $\epsilon>0$ there is an L such that for each f in the collection we have $|f(x)|<L$ except on a set I the measure of whose projection on any r-dimensional coordinate subspace is less than ϵ.

The author considers a bounded domain T in Euclidean n-space, whose boundary Γ is formed from a finite number of regular varieties with local representations which are differentiable and whose normal vectors have direction cosines Lipschitzian in the local coordinates. He defines a class $\mathfrak{B}^{(\alpha)}$ of functions $u(x)$ on T, satisfying α) $u(x)$ is continuous, β) $u(x)$ and its partial derivatives $p_i(x)=\partial u/\partial x_i$ (supposed to exist almost everywhere) are absolutely continuous with respect to x_i for almost all choices of $(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$, and γ) the second derivatives $p_{ij}(x)=\partial^2u/\partial x_i\partial x_j$ (which, from β), exist almost everywhere and are measurable) are in L_α ($\alpha\geqq 1$). He recalls a definition from another paper [Ricerche Mat. **1**, 27–54 (1952); these Rev. **14**, 30]: A function $f(x)$ is of class $\mathfrak{A}^{(\alpha)}$ in T if $f(x)$ is absolutely continuous in x_i for almost all $(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$ and the first partial derivatives of $f(x)$ are in L_α on T.

For convenience in summarizing the results the reviewer introduces the notation

$$\|u\|^{(\alpha)}=\int_T\left[|u(x)|^\alpha+\sum_{i=1}^n|p_i(x)|^\alpha+\sum_{i,j=1}^n|p_{ij}(x)|^\alpha\right]dx.$$

The author first states a compactness theorem: (I) if the sequence $\{u_m\}$ of functions in $\mathfrak{B}^{(\alpha)}$ with $\alpha>1$ satisfies $\|u_m\|^{(\alpha)}\leqq A$ for all m, there is for any integer $r<\alpha$ a subsequence converging μ_{n-r}-quasi uniformly along with the sequence of first partial derivatives, on the interior of T. In the case $\alpha=2$, if the original sequence is in addition equi-μ_{n-2}-quasi bounded in T, the convergence may be taken to be μ_{n-2}-quasi uniform. This follows easily from results in the author's paper cited above. The second theorem concerns boundary values. Let $x_i=x_i(\xi_1, \cdots, \xi_{n-1})$ be a representation of Γ. Let $\mu(\xi)$ be a continuous vector function for ξ on Γ, representing a vector beginning at the point ξ on Γ and pointing into T; further, $\mu(\xi)$ is assumed never tangent to Γ. Then: (II) if $f(x)$ is of class $\mathfrak{A}^{(2)}$ in T, almost everywhere on Γ there exists the limit $\lim_{\rho\to 0} f[x(\xi)+\rho\mu(\xi)]$. Calling this $\phi(\xi)$, it turns out that ϕ is in L_2 on Γ. Also, ϕ is independent of the choice of the functions $\mu(\xi)$. This extends a result of Fichera [Ann. Scuola Norm. Super. Pisa (3) **4**, 35–99 (1950), p. 78; these Rev. **13**, 42].

For $u \in \mathfrak{B}^{(2)}$, put

$$E(u)=\sum_{i,j=1}^n A_{ij}(x)p_{ij}(x)+\sum_{i=1}^n B_i(x)p_i(x)+C(x)u(x),$$

where the A_{ij}'s are $C^{(2)}$, the B_i's $C^{(1)}$, and C continuous. The A_{ij}'s are assumed always to be the coefficients of a positive definite quadratic form. Let $f(x)$ be continuous and in L_2 on T. We are concerned with the equation (1) $E(u)=f$. Let $\phi(x)$ for $x \in \Gamma$ be in $\mathfrak{B}^{(2)}$ in the local coordinates for Γ, i.e., $\phi(x(\xi_1, \cdots, \xi_{n-1}))$ is $\mathfrak{B}^{(2)}$ in the space of the ξ's. Then the author proves that (III) if $C(x)\leqq 0$ throughout, then (1) has a solution in $\mathfrak{B}^{(2)}$ taking on the boundary values of $\phi(x)$. Further, he extends an integral estimate of Cacciopoli [Giorn. Mat. Battaglini (4) **4**(**80**), 186–212 (1951), p. 202; these Rev. **13**, 749] to read: (IV) if $u \in \mathfrak{B}^{(2)}$ and if u takes

on boundary values given by a function ϕ as above, then

$$\|u\|^2 \leqq \text{const.} \times \left\{ \int_T [E(u)]^2 dx + \int_\Gamma \phi^2 d\sigma + \sum_{i,j=1}^{n-1} \int_\Gamma \frac{\partial^2 u}{\partial \xi_i \partial \xi_j} d\sigma \right\}.$$

Turning to variational problems in which the integrand involves the second derivative, he considers functions $F(x; u; p_1, \cdots, p_n; p_{n}, \cdots, p_{nn})$ for $x \in T$ and the other variable arbitrary. The Weierstrass $\mathcal{E}$-function is

$$\begin{aligned} \mathcal{E}(x; u; p_1, \cdots, p_n; p_{11}, \cdots, p_{nn}; \bar{p}_{11}, \cdots, \bar{p}_{nn}) \\ = F(x; u; p_1, \cdots, p_n; p_{11}, \cdots, p_{nn}) \\ - F(x; u; p_1, \cdots, p_n; \bar{p}_{11}, \cdots, \bar{p}_{nn}) \\ - \sum (p_{ij} - \bar{p}_{ij}) F_{p_{ij}}(x; u; p_1, \cdots, p_n; \bar{p}_{11}, \cdots, \bar{p}_{nn}). \end{aligned}$$

We assume this always non-negative. Further, assume that for any $u \in \mathfrak{B}^{(2)}$ we have

$$F(x; u; p_1, \cdots, p_n; p_{11}, \cdots, p_{nn}) \geqq [E(u)]^2 + P,$$

where $E(u)$ is as before and P is a constant. Let $\bar{\mathfrak{B}}^{(2)}$ be the subclass of $\mathfrak{B}^{(2)}$ whose elements u yield finite values to the integral

$$I(u) = \int_T F(x; u(x); p_1(x), \cdots, p_n(x); p_{11}(x), \cdots, p_{nn}(x)) dx$$

and take on boundary values given by $\phi(x)$, $x \in \Gamma$, where ϕ is as before. Then, using his integral estimate, the sharpened form for $\alpha = 2$ of his compactness theorem, and employing a semicontinuity argument due to Tonnelli, the author (V) proves the existence of a minimizing function in $\bar{\mathfrak{B}}^{(2)}$.

The remainder of the paper considers the Euler equations of minimum problems under various conditions. As an example, he proves that the equation $\Delta_4 u = f(x_1, \cdots, x_n)$ has a solution among those functions u of $\mathfrak{B}^{(2)}$ which satisfy $u(x) = \alpha(x)$ and $du(x)/dn = \beta(x)$ on the boundary Γ (in the sense of the boundary-value theorem (I) quoted above), where α and β are of class $\mathfrak{B}^{(2)}$ on Γ in the local coordinates.

J. M. Danskin (Washington, D. C.).

Theory of Probability

Richter, Hans. Zur Grundlegung der Wahrscheinlichkeitstheorie. I. Vergleichende Betrachtung bestehender Theorien. Math. Ann. **125**, 129–139 (1952).

To prepare the ground for an axiomatic foundation of probability to be published as Part II, the author devotes the present paper to a review of some of the existing treatments of the subject. He dwells on their common ground and some of the issues they raise, rather than their details. His conclusions are in part as follows.

Every theory of probability must contain a measure-theoretical structure; and, also, rules for relating the measure with experiment. Explicit reference must be made to the experimental procedure underlying the "events" considered, as well as for the combination of different experimental procedures. Only after this can such principles as compound probability be formulated. Probability cannot be derived from any other branch of science, but has to be defined implicitly by ad hoc axioms. The objective point of view has to be supplemented by a subjective one. Vice versa, subjectivists formulate processes by which new data "improve" their evaluations of the probability of a given event, and hence seem to have a "true value" of its probability in mind, apparently an objective ideal.

The chief originality is in the presentation.

B. O. Koopman (New York, N. Y.).

van der Waerden, B. L. Der Begriff Wahrscheinlichkeit. Studium Generale **4**, 65–68 (1951).

The Laplace, von Mises, and similar conceptions of probability are reviewed briefly, and rejected for the usual reasons. In rejecting all subjective approaches, the author fails to distinguish between those which purport to define probability and give rules for finding its values ad hoc, and those which attempt merely to give rules by which certain probabilities can be deduced from others (rules of consistency). The usual invalidating arguments apply to the former but not to the latter. The fact that probability is in principle undefinable is well realized and stated by the author. He then adopts the Kolmogoroff measure-axioms, and, following Kolmogoroff, separates from these the application problem, which he regards as a philosophical one. He considers that rough practical rules can take care of the latter, as far as experimental science is concerned. He lists such rules as that probability is to be measured by frequency; a very improbable event will not happen; under practically similar conditions the same event will have the same probabilities; if two events have no causal connection, they are independent. The author seems unaware of the fundamental distinction between application rules in probability and those in any other science: In mechanics, the rules involve, in last analysis, some concept of probability (since they accept as "practically certain" many statements that are not logically proved). To base the application theory of probability on such analogy would involve an essential circularity. The author does not make it clear that the difficulties he noted earlier will not reappear in his application theory. *B. O. Koopman* (New York, N. Y.).

Vietoris, Leopold. Wie kann Wahrscheinlichkeit definiert werden? Studium Generale **4**, 69–72 (1951).

The issues raised by the attempt to define a scientific term are discussed, in general and as applied to probability. The impossibility of obtaining an explicit definition of probability by such schemes as the Laplace and frequency methods is pointed out, with the conclusion that the impossibility of defining probability explicitly is an underlying principle: It must be defined implicitly by axioms. When this task is undertaken, there are advantages, as the author shows, in using as the fundamental undefined notion that of comparison in probability, rather than the numerical value of probabilities. The author refers to his own and the reviewer's systems of axioms. *B. O. Koopman.*

Gröbner, W. Über die Anwendung des Wahrscheinlichkeitsbegriffes in der Physik. Studium Generale **4**, 72–77 (1951).

The author discusses, in general terms, the place given to probability in physics and, in particular, in quantum mechanics. He insists on keeping clear the distinction between a priori probability (axiomatized, with a more or less subjective connotation) and a posteriori probability (observed, as in statistical frequencies). He objects to the frequentists' elimination of the former. He goes to lengths in objecting to the various confused general conclusions about probability and causality, frequently drawn by quantum theory enthusiasts who force everything into the frame of extreme positivism. While some of their contradictions are mentioned, no mathematical discussion is given.

B. O. Koopman (New York, N. Y.).

Morlat, Georges. Sur une généralisation de la loi de Poisson. C. R. Acad. Sci. Paris **235**, 933–935 (1952).

The generalization in question can be regarded as a very special case of renewal theory. *K. L. Chung.*

Morlat, Georges. Sur une classe de fonctions aléatoires. C. R. Acad. Sci. Paris **235**, 1364–1366 (1952).

The author studies the process in which $x(t)$ takes the values 1 and 0 alternatively in successive intervals whose lengths are independent and distributed according to the laws $F(v)$ and $G(v)$ alternatively. It is difficult to judge the results from the abstract, but if $\Xi_n(t)$ is $n^{-1/2}$ times the sum of n independent and identically distributed two-valued random variables with mean 0, as it appears, it is not clear how $\Xi_n(t)$ can converge in any significant way to a random variable as $n\to\infty$. *K. L. Chung* (Ithaca, N. Y.).

Putnam, Calvin R., and Wintner, Aurel. On the addition of symmetric normal frequency curves. Math. Notae **11**, 79–86 (1951).

The authors consider the problem whether or not the probability distribution over $(-\infty, \infty)$ with density

$$f(x)=\pi^{-1/2}\sum_k p_k\alpha_k^{1/2}\exp(-\alpha_k x^2)\quad(\sum_k p_k=1,\ p_k>0)$$

is bell-shaped. They deal in detail with the case where the sum defining $f(x)$ has only two terms, and show that for some choices of the parameters $f(x)$ has three points of inflexion in $(0, \infty)$, while for other choices $f(x)$ has only one such point (and hence has a bell-shaped graph). [See also the following review.] *H. P. Mulholland* (Birmingham).

Levi, Beppo. On the form of composite frequency curves. Math. Notae **11**, 87–109 (1951). (Spanish)

The author's paper is a sequel to that considered in the preceding review: the notation used there will be retained. Suppose that k runs from 1 to m (finite), and that n denotes the number of points of inflexion of $f(x)$ in $(0, \infty)$. When $m=2$, the author shows that $n=1$ if the ratio of the least of the numbers α_k to the greatest is not less than $5-\sqrt{24}$, but that otherwise n can be either 1 or 3. When $m=2, 3, \cdots$, he shows that n must be odd and such that $1\leqq n\leqq 2m-1$. For $m=3$ he gives a numerical example in which $n=5$. *H. P. Mulholland* (Birmingham).

Johnson, N. L. Approximations to the probability integral of the distribution of range. Biometrika **39**, 417–419 (1952).

An approximation formula is given for the probability $P_n(w)$ that the range of n independent random variables does not exceed w. The approximation is good for small values of w and n not too large. Approximations are also given for the significance limits w_α satisfying $P_n(w_\alpha)=\alpha$. *E. Lukacs* (Washington, D. C.).

Lukacs, Eugene, and Szász, Otto. On analytic characteristic functions. Pacific J. Math. **2**, 615–625 (1952).

A sufficient condition for the existence of moments is given. This involves the difference quotients of the characteristic function at 0; for odd moments cf. a result by Zygmund [Ann. Math. Statistics **18**, 272–276 (1947); these Rev. **9**, 88]. Some known theorems on analytic characteristic functions are proved simply. Using a theorem of Lévy and Raikov [see Dugué, Ann. Inst. H. Poincaré **12**, 45–56 (1951); these Rev. **12**, 838], it is shown that the characteristic function of an infinitely divisible law has no zero inside its strip of convergence; as a result it is shown that the infinitely divisible characteristic function $a^2/(a^2+t^2)$ can be decomposed into two factors neither of which is infinitely divisible. *K. L. Chung* (Ithaca, N. Y.).

Robbins, Herbert. A note on gambling systems and birth statistics. Amer. Math. Monthly **59**, 685–686 (1952).

Let $\{x_n\}$ be a sequence of independent random variables with $E\{x_n\}=0$ and interpret x_n to be a gambler's gain in the nth play of a game. Then, under suitable conditions, a system of stopping having a finite expected stopping time leads to an expected gain of 0. Blackwell [Ann. Math. Statistics **17**, 84–87 (1946); these Rev. **8**, 478] showed that it is sufficient to assume that $|x_n|\leqq C$ for all n. The author shows that it is sufficient to assume that $E|x_n|\leqq C$ for all n. An application to birth statistics when one assumes that parents have boys with probability $\frac{1}{2}$ but tend to stop after the first boy is discussed. *J. L. Snell* (Princeton, N. J.).

Lévy, Paul. Loi faible et loi forte des grands nombres. C. R. Acad. Sci. Paris **235**, 1186–1188 (1952).

Let X_ν be independent random variables with median 0 unless otherwise specified; $S_n=\sum_1^n X_\nu$, $m_n=$median of S_n, $M_n=\mathrm{Max}_{\nu\leq n}|X_n|$. Let $U_n=o_f(V_n)$ and $U_n=o_F(V_n)$ denote, respectively, U_n/V_n tends to 0 in probability and with probability one. Let a_n be positive numbers increasing to infinity. Consider the relations (1_f) $S_n-m_n=o_f(a_n)$, (2_f) $M_n=o_f(a_n)$ and similarly (1_F) and (2_F). (1_f) and (1_F) imply respectively (2_f) and (2_F). For the converse there are three cases: (a) Laplacian: S_n tends in distribution to the normal in such a way that $\Pr(M_n<\epsilon|S_n-m_n|)\to 1$ for every $\epsilon>0$; (b) intermittently Laplacian; (c) (uniformly) non-Laplacian. Only in the last case does (2_f) imply (1_f) for every sequence a_n specified above. Thus (1_f) and (2_f) are equivalent in the non-Laplacian case if $m(X_\nu)=o(a_\nu)$. Further, let $F_\nu(x)=\Pr(|X_\nu|>x)$ and $x^2F_\nu(x)/\int_0^x y^2|dF_\nu(y)|=w_\nu(x)$. If $w_\nu(x)$ exceeds a positive constant independent of ν, we are in the "equally non-Laplacian" case. Then (1_F) and (2_F) are equivalent if $m(X_\nu)=o(a_\nu)$. If all $F_\nu(x)=F(x)$, then (1_F) and (2_F) are equivalent if and only if $w(x)$ exceeds a positive constant, without any condition on the median. This result is to be compared with a result of Feller [Amer. J. Math. **68**, 257–262 (1946); these Rev. **8**, 37], which imposes regularity conditions on a_ν but a weaker condition on $F(x)$. According to a letter from the author, there is another interesting case, called "uniformly Laplacian", in which $w_\nu(x)$ is independent of ν and $\to 0$ as $x\to\infty$. No proofs are given. *K. L. Chung* (Ithaca, N. Y.).

Yaglom, A. M. Introduction to the theory of stationary random functions. Uspehi Matem. Nauk (N.S.) **7**, no. 5(51), 3–168 (1952). (Russian)

Expository paper, with only indications of proofs but with an extensive set of references. Particular stress is laid on prediction and filtering problems, especially those involving stationary stochastic processes with a discrete [continuous] parameter having spectral densities which are rational functions of $e^{i\lambda}$ [λ]. A great many examples are discussed in detail. *J. L. Doob* (Urbana, Ill.).

Matschinski, Matthias. Quelques remarques sur les processus stochastiques. Le processus stochastique dans une population. C. R. Acad. Sci. Paris **235**, 1362–1364 (1952).

An attempt to limit the discussion of certain stochastic processes by requiring probabilities to satisfy a frequency definition. *J. L. Snell* (Princeton, N. J.).

Uberoi, Mahinder S., and Kovasznay, Leslie S. G. On mapping and measurement of random fields. Quart. Appl. Math. **10**, 375–393 (1953).

Lorsqu'on mesure une grandeur $U(X)$ scalaire ou vectorielle attachée à chaque point X d'un domaine spatial, l'instrument de mesure fournit une grandeur $\Omega(X)$ différente de $U(X)$ et qui est liée à $U(X)$ par une relation fonctionnelle linéaire. Les auteurs mettent cette relation sous la forme

$$\Omega(x)=\int K(X-S)U(S)dV(S),$$

K étant un noyau caractéristique de l'instrument de mesure, et $dV(S)$ l'élément de volume au point S. Ils étudient les renseignements qui fournit sur U la connaissance de Ω, lorsque U est un champ aléatoire homogène, par exemple, la vitesse ou la densité d'un fluide turbulent. Ils établissent les relations entre les fonctions de corrélation et fonctions spectrales mesurées sur Ω et celles de U. Deux exemples sont traités en détails: (1) Mesures, à l'aide d'un anémomètre à fil chaud, de longueur $2l$, de la vitesse d'un fluide turbulent, isotrope et incompressible. On peut choisir

$$K(S)=\begin{cases}\delta(s_1)\delta(s_2) & \text{si } |s_3|\leqq l\\ 0 & \text{si } |s_3|>l.\end{cases}$$

Relations entre les fonctions de corrélation longitudinales de vitesse et les spectres théoriques et mesurés. Cas où l est grand par rapport à l'échelle de la turbulence. (2) Mesures du champ de densités par la méthode des ombres. Le courant turbulent, de largeur $2l$, est traversé par un flux lumineux qui impressionne une plaque photographique. La variation relative d'intensité h sur la plaque est liée à la densité ρ par la formule linéaire

$$h(x_1, x_2, x_3)=\int_{x_3-l}^{x_3+l}\left(\frac{\partial^2\rho}{\partial x_1^2}+\frac{\partial^2\rho}{\partial x_2^2}\right)dx_3;$$

h correspond à Ω et ρ à U dans la formule générale.

J. Bass (Berkeley, Calif.).

Hollingsworth, C. A. Solutions of some problems concerning long random coils. J. Chem. Phys. **20**, 1580–1590 (1952).

A particle performs a discrete random flight in two or more dimensions with Gaussian transition probability. Assuming that the number N of steps in the random flight is very large, approximate solutions are derived for ten probability problems related to the probability that the flight passes within a given small distance from one or more given points. Typical example: "Determine the probability that a random flight of N steps starting from the origin passes at least once through a volume element at a given point P and ends in a volume element at a given point R". Some of these probabilities are obtained by solving the corresponding boundary-value problem for the heat equation; the others then follow by general consideration of conditional probabilities. Physical interpretations in terms of diffusion, heat flow, and polymer theory are discussed. *W. Wasow.*

Mathematical Statistics

David, H. A. An operational method for the derivation of relations between moments and cumulants. Metron **16**, nos. 3–4, 41–47 (1952).

Kaplan, E. L. Tensor notation and the sampling cumulants of k-statistics. Biometrika **39**, 319–323 (1952).

Gini, Corrado. L'evoluzione del concetto di media. Metron **16**, nos. 3–4, 3–26 (1952).

Quensel, Carl-Erik. The distribution of the second order moments in random samples from non-normal multivariate universes. Lunds Univ. Årsskrift. N. F. Avd. 2. **48** = Kungl. Fysiog. Sällskapets Handlingar. N. F. **63**, no. 4, 11 pp. (1952).

The author is interested in procedures and general techniques for obtaining the distributions of all second order moments from a multivariate Gram-Charlier Type A series. He determines the characteristic function of the second order central moments and indicates in some special cases the results for the distribution functions themselves.

L. A. Aroian (Culver City, Calif.).

Simpson, E. H. The interpretation of interaction in contingency tables. J. Roy. Statist. Soc. Ser. B. **13**, 238–241 (1951).

The definition of second order interaction in a $(2\times2\times2)$ table given by Bartlett is accepted, but it is shown by an example that the vanishing of this second order interaction does not necessarily justify the mechanical procedure of forming the three component 2×2 tables and testing each of these for significance by standard methods. (Author's summary.) *H. Chernoff* (Stanford, Calif.).

Lancaster, H. O. Complex contingency tables treated by the partition of χ^2. J. Roy. Statist. Soc. Ser. B. **13**, 242–249 (1951).

The partition of χ^2 can be used to investigate the interactions of all orders in the higher contingency tables. The χ^2, obtained by partition, in the case of the $2\times2\times2$ table for the second order interaction is asymptotically equal to that obtained by Bartlett [J. Roy. Statist. Soc. Suppl. **2**, 248–252 (1935)]. Difficulties arise when an attempt is made to find an exact solution. The method of obtaining matrices for orthogonal transformations of variables arranged in hierarchical order is explained. The orthogonal transformation in the case of the $2\times2\times2$ table is given in full. (Author's summary.) *H. Chernoff* (Stanford, Calif.).

Bruno, Angelo. Valor medio della potenza dello scarto di una variabile casuale nelle prove ripetute. Boll. Accad. Gioenia Sci. Nat. Catania (4) no. **8**, 520–528 (1951).

The moments about the mean $E\{(X-np)^k\}$ of the binomial distribution are expressed in terms of numbers a_{rs} defined recursively by $a_{r1}=a_{rr}=1$ for $r=1, 2, \cdots, r$ and by $a_{rs}=sa_{r-1}+a_{r-1,s-1}$ for $r=3, 4, \cdots$; $s=2, 3, \cdots, r-1$. Some observations on approximate values for these moments are made. *Z. W. Birnbaum* (Seattle, Wash.).

Masuyama, Motosaburo. An approximation to the noncentral t-distribution with the stochastic paper. Rep. Statist. Appl. Res. Union Jap. Sci. Eng. **1**, no. 3, 28–31 (1951).

The author gives a further discussion of his improved binomial probability paper [same Rep. **1**, no. 2, 15–22 (1951); these Rev. **13**, 961]. It is shown that the paper can be used in place of the approximation of Johnson and Welch to the noncentral t-distribution [Biometrika **31**, 362–389 (1940); these Rev. **1**, 346]. For example, if z has the normal distribution $N(0, 1)$ and fw has a chi-square distribution with

f degrees of freedom and is independent of z, the equation Prob $[(z+\delta)/\sqrt{w}>t]=\epsilon$ can be solved for δ if f, ϵ, and t are specified. *T. E. Harris* (Santa Monica, Calif.).

Masuyama, Motosaburo. Revision of the tables in "An improved binomial probability paper and its use with tables". Rep. Statist. Appl. Res. Union Jap. Sci. Eng. 1, no. 3, 32–33 (1951).

A revision is given for two of the tables in the article on binomial probability paper referred to in the preceding review. The changes are small percentagewise.

T. E. Harris (Santa Monica, Calif.).

Hyrenius, Hannes. Sampling from bivariate non-normal universes by means of compound normal distributions. Biometrika 39, 238–246 (1952).

The author is interested in the distributions of the sample means, variances, and covariances from general bivariate compound normal distributions. Let

$$\varphi(x_1, x_2)=\varphi(x_1, x_2; \alpha_1, \alpha_2, \sigma_1^2, \sigma_2^2, \rho),$$

the bivariate normal distribution with means α_1, α_2; variances σ_1^2, σ_2^2, and coefficient of correlation ρ. If the subscript i indicates a bivariate normal distribution, the bivariate compound normal distribution is defined by (*) $f(x_1, x_2)=\sum_{i=1}^n p_i\varphi_i(x_1, x_2)$. The moments and characteristic function of the first and second order moments when sampling from (*) are derived. The author then considers the case where α_{1i} and α_{2i} vary, but $\sigma_{1i}=\sigma_1$, $\sigma_{2i}=\sigma_2$, $\rho_i=\rho$, and for this case derives the joint distribution of the sample means, sample variances and covariance, and the joint distribution of the sample variances and covariance, as well as the individual distributions of these statistics. Applications are made to the distribution of the coefficient of correlation from (*) and the regression coefficient in linear regression. Numerical examples of the theory and the case of varying second moments will appear in later papers.

L. A. Aroian (Culver City, Calif.).

Isida, Masatugu D. A remark on the linear regression estimate. Ann. Inst. Statist. Math., Tokyo 4, 7–9 (1952).

A random sample of size n is drawn from an infinite population in which the regression of y on x is linear. Approximate expressions, valid for large n, are given for the 3rd and 4th moments of the frequency distribution of the linear regression estimate $\bar{y}_L$ of the population mean of y, where $\bar{y}_L=\bar{y}+b(\bar{X}-\bar{x})$ and $\bar{y}$, $\bar{x}$, are the sample means of y, x, respectively, $\bar{X}$ is the population mean of x and b is the least squares regression coefficient of y on x. The author also shows, by non-rigorous methods, that all moments of $\bar{y}_L$ tend to those of the normal distribution as $n\to\infty$.

W. G. Cochran (Baltimore, Md.).

Chanda, K. A note on the comparative efficiencies of selection of sampling units with and without replacement. Science and Culture 18, 288–289 (1952).

Aoyama, Hirojiro. On practical systematic sampling. Ann. Inst. Statist. Math., Tokyo 3, 57–63 (1952).

A population contains N sampling units ($N=kn-r$, where k and n are integral and $0\leqq r<k$). A systematic sample of size n is drawn by choosing a random number between 1 and k, and taking every kth unit thereafter. The sample mean is shown to be a biased estimate of the population mean if $r\neq 0$, or if some of the observations in the sequence do not belong to the population. Expressions for the biases are given. *W. G. Cochran* (Baltimore, Md.).

Masuyama, Motosaburo. A graphical method of estimating parameters in Kapteyn distributions. Rep. Statist. Appl. Res. Union Jap. Sci. Eng. 1, no. 4, 32–34 (1952).

Grundy, P. M. The fitting of grouped truncated and grouped censored normal distributions. Biometrika 39, 252–259 (1952).

The maximum likelihood estimation of the parameters of a truncated or censored normal distribution were given by A. Hald [Skand. Aktuarietidskr. 32, 119–134 (1949); these Rev. 12, 193]. A distribution is said to be censored if the frequency of observations below the known truncation point is known although the observations are not recorded. Gjeddebæck [ibid. 32, 135–159 (1949); these Rev. 11, 446] considered the effect of grouping and gave maximum likelihood estimates for the population parameters. The author deals with the same problem by introducing adjusted sample moments and obtains his estimates using the adjusted moments and Hald's tables for the ungrouped distribution.

E. Lukacs (Washington, D. C.).

Gupta, A. K. Estimation of the mean and standard deviation of a normal population from a censored sample. Biometrika 39, 260–273 (1952).

The author observes that there are two different ways of censoring a sample: (1) observations below or above a given truncation point may be censored (type I) [see the preceding review and references given there]; (2) the $n-k$ smallest or greatest observations out of a sample of size n may be censored (type II). The paper deals with the problem of estimating mean and standard deviation of a normal population from a censored sample of type II. Several tables are given, one for the computation of maximum likelihood estimates of the population parameters, another for obtaining the asymptotic variances and covariances of these estimates. Finally linear estimates in small samples are discussed and tables are given for the computation of these estimates. *E. Lukacs* (Washington, D. C.).

Cox, D. R. Estimation by double sampling. Biometrika 39, 217–227 (1952).

An approximate large-sample solution is given for the problem of estimating a parameter θ with specified variance $a(\theta)$, using a two-stage sequential sampling plan. Applications. *J. L. Hodges, Jr.* (Berkeley, Calif.).

Kiefer, J. Sequential minimax estimation for the rectangular distribution with unknown range. Ann. Math. Statistics 23, 586–593 (1952).

The author considers the problem of sequential estimation assuming that the unknown distribution has a density function of the form $f(y;\theta)=1/\theta$, $0<y<\theta$, where $0<\theta<\infty$. The cost is assumed constant per sample and the loss if the estimate d is made when θ is correct is $((\theta-d)/\theta)^2$. Under these conditions a procedure requiring a fixed sample size is shown to be a minimax solution. *J. L. Snell.*

Anscombe, F. J. Large-sample theory of sequential estimation. Proc. Cambridge Philos. Soc. 48, 600–607 (1952).

Let $\{Y_n\}$ ($n=1, 2, \cdots$) be an infinite sequence of random variables. Suppose that there exist a real number θ, a sequence of positive numbers $\{w_n\}$, and a distribution function

$\mathfrak{F}(x)$, such that the following conditions are satisfied: For any x such that $\mathfrak{F}(x)$ is continuous, $\text{Prob}(Y_n-\theta\leqq xw_n)\rightarrow\mathfrak{F}(x)$ as $n\rightarrow\infty$, and given any small positive ϵ and η, there is a large ν and small positive c such that, for any $n>\nu$, $\text{Prob}\{|Y_{n'}-Y_n|<\epsilon w_n$ uniformly for all integers n' such that $|n'-n|<cn\}>1-\eta$. Now let $\{n_r\}$ be a monotonic sequence of positive integers tending to infinity, and let $\{N_r\}$ be a sequence of random variables taking on positive integral values such that $N_r/n_r\rightarrow 1$ in probability as $r\rightarrow\infty$. Then $\text{Prob}\{Y_{N_r}-\theta\leqq xw_{n_r}\}\rightarrow\mathfrak{F}(x)$ as $r\rightarrow\infty$ at all continuity points x of $\mathfrak{F}(x)$. The preceding is the main result of the paper. The author uses this theorem to determine a sequential stopping rule giving prescribed accuracy in cases of sequential point estimation. Several examples are given and some related topics discussed. *R. P. Peterson.*

Whittle, P. The simultaneous estimation of a time series harmonic components and covariance structure. Trabajos Estadística 3, 43–57 (1952). (Spanish summary)

The author discusses estimation problems for a time series composed of the sum of a (purely non-deterministic) stationary series and a deterministic component which is a finite sum of harmonic terms. The spectral density of the stationary series depends on p parameters. Estimates of the unknown parameters and frequencies are indicated, as well as sampling distributions of these estimates. The reasoning is sketchy, and no precise formulations or proofs are given. *J. L. Doob* (Urbana, Ill.).

Petrov, A. A. The verification of an hypothesis concerning the normality of distributions by small samples. Translated by C. D. Benster. National Bureau of Standards, Washington, D. C., Rep. 2116, 9 pp. (1952).

Translated from Doklady Akad. Nauk SSSR (N.S.) 76, 355–358 (1951); these Rev. 12, 622.

Matusita, Kameo. Correction to the paper, "On the theory of statistical decision functions." Ann. Inst. Statist. Math., Tokyo 4, 51–53 (1952).

See same Ann. 3, 17–35 (1951); these Rev. 13, 668.

Sandelius, Martin. A confidence interval for the smallest proportion of a binomial population. J. Roy. Statist. Soc. Ser. B. 14, 115–116 (1952).

After finding the distribution of the smaller relative frequency of a binomial population, the author uses the general method of finding confidence intervals, described, e.g., by A. M. Mood [Introduction to the theory of statistics, McGraw-Hill, New York, 1950, §11.5; these Rev. 11, 445], to construct the desired confidence interval for the smaller proportion of a binomial population. *S. W. Nash* (Vancouver, B. C.).

Chapman, Douglas G. On tests and estimates for the ratio of Poisson means. Ann. Inst. Statist. Math., Tokyo 4, 45–49 (1952).

The Przyborowski-Wilenski test of the equality of two Poisson means μ and $\lambda\mu$ is extended to provide (shortest unbiased) confidence intervals for the ratio λ. There is no unbiased estimate for λ of finite variance, but an estimate is proposed whose bias is only $\lambda e^{-\mu}$. *J. L. Hodges, Jr.*

Drion, E. F. Some distribution-free tests for the difference between two empirical cumulative distribution functions. Ann. Math. Statistics 23, 563–574 (1952).

Let $F_1(x)$ and $F_2(x)$ be the empirical distribution functions of two independent sets of n_1 and n_2 independent chance variables with a common continuous distribution function. The probability that either $F_1(x)-F_2(x)<0$ or $F_1(x)-F_2(x)>0$ for all x between the least and the largest observation is found to be $1/(n_1+n_2-1)$ if $n_1=n_2$, $2/(n_1+n_2)$ if n_1 and n_2 are coprime, and less than $2/(n_1+n_2)$ in all other cases. Other results concerning the exact distribution of $\max_x|F_1(x)-F_2(x)|$ when $n_1=n_2$ are identical with those of Gnedenko and Korolyuk [Doklady Akad. Nauk SSSR (N.S.) 80, 525–528 (1951); these Rev. 13, 570]. *W. Hoeffding* (Chapel Hill, N. C.).

David, F. N., and Johnson, N. L. Extension of a method of investigating the properties of analysis of variance tests to the case of random and mixed models. Ann. Math. Statistics 23, 594–601 (1952).

Results are given whereby the methods described in an earlier paper [same Ann. 22, 382–392 (1951); these Rev. 13, 143], dealing with the parametric case, may be applied also to the case of random, or mixed random and parametric components. (Author's summary.) *L. J. Savage.*

Scheffé, Henry. An analysis of variance for paired comparisons. J. Amer. Statist. Assoc. 47, 381–400 (1952).

In a paired comparison test of m brands each possible pair is compared by $2r$ judges, r for each order of presentation. Account is taken of the order of presentation, and main effects are defined for the brands. An analysis of variance is worked out. The hypothesis of subtractivity states that the results for any pair of brands, once order effects have been eliminated, is due entirely to main effects of the two brands. Significance tests are given for the main effects, for the order effects, and for the hypothesis of subtractivity. The author urges use of the Studentized range for making comparisons of main effects. Least squares estimates of various parameters and their standard errors are given. *S. W. Nash* (Vancouver, B. C.).

Walsh, John E. Some nonparametric tests for Student's hypothesis in experimental designs. J. Amer. Statist. Assoc. 47, 401–415 (1952).

Let

$$y_{ijk}=\mu+\tau_i+\beta_j+e_{ijk}\qquad(k=1,\cdots,u_{ij};\ i=1,\cdots,m;\ j=1,\cdots,n)$$

where μ, τ_i, β_j are parameters and e_{ijk} is a random variable with zero mean. It is desired to test whether a linear combination $\sum w_i\tau_i$ of the τ_i (where $\sum w_i=0$) is zero or to obtain a confidence interval for $\sum w_i\tau_i$. Let $\bar{y}_{ij}=\sum_k y_{ijk}/u_{ij}$, $Y_j=\sum_i w_i\bar{y}_{ij}$. Under assumptions which ensure that the Y_j are mutually independent and each Y_j has a symmetrical and continuous distribution, tests and confidence intervals for $\sum w_i\tau_i$ are obtained which are based on previous results of the author [Ann. Math. Statistics 20, 64–81 (1949); these Rev. 10, 554]. *W. Hoeffding* (Chapel Hill, N. C.).

Williams, R. M. Experimental designs for serially correlated observations. Biometrika 39, 151–167 (1952).

The experimental units in a controlled experiment (e.g., the plots in an agricultural experiment) are arranged in some linear sequence. The observation on the ith unit is y_i $(0\leqq i\leqq n)$, where $y_i=x_i+a_{[i]}$. The symbol $a_{[i]}$ represents the effect of the treatment applied to the ith unit. If the error component x_i follows a stationary Markoff process, $x_i=\rho x_{i-1}+\epsilon_i$, $|\rho|<1$, where $\epsilon_i\sim N(0,\sigma^2)$, independently of x_{i-1} and all preceding x's, the author shows that suitable experimental designs, from the point of view of precision

and of ease in computing the maximum likelihood estimates of the $a_{[i]}$, have the properties that (A) each treatment occurs equally often adjacent to every other treatment, or (B) each treatment occurs equally often adjacent to every treatment, including itself. Examples of such designs are presented. For large ρ, positive or negative, these designs are shown to be more precise than randomized blocks.

For the second order Markoff process,

$$x_i+d_1x_{i-1}+d_2x_{i-2}=\epsilon_i,$$

a suitable design requires the additional condition that each treatment occur equally often adjacent but one to every other treatment. Goodness-of-fit tests of these models are made on two series of agricultural plot data.

W. G. Cochran (Baltimore, Md.).

Dalenius, Tore. The problem of optimum stratification in a special type of design. Skand. Aktuarietidskr. **35**, 61–70 (1952).

A finite population contains ν sampling units. The values of some variate y on these units are arranged in increasing order, and the population is to be divided into 2 strata at the point y_0. A stratified random sample of size n is drawn, subject to the condition that the sample in the upper stratum ($y>y_0$) contains all units in the stratum. The value of y_0 which minimizes the variance of $\bar{y}_{st}$, the usual estimate of the population mean from a stratified random sample, is shown to satisfy the equation

$$(y_0-\mu_1)^2=\frac{p_1\sigma_1^2}{p_1-1+f}$$

where p_1 is the proportion of the population lying in the lower stratum, μ_1, σ_1^2 are the mean and variance in the lower stratum, and $f=n/\nu$ is the sampling fraction. The author compares the variance of $\bar{y}_{st}$ with the variance given by the usual optimum allocation for fixed n, for the frequency distributions e^{-y}, ye^{-y} ($y\geqq 0$) and for the Swedish income distribution. These illustrations indicate, as would be expected intuitively, that the restricted stratification is inefficient unless f is substantial, say 0.25.

W. G. Cochran (Baltimore, Md.).

Malatesta, Sante. Contributo allo studio statistico delle communicazioni. Alta Frequenza **21**, 163–198 (1952).

Si forniscono i primi elementi per lo studio dei circuiti con metodi statistici. Lo studie è compiuto mediante due particolari funzioni, la densità spettrale e la funzione di correlazione, e mediante un criterio per valutare l'efficienza dei circuiti in base alla distorsione. Viene mostrata l'applicazione del metodo al filtro ottimo di rumore di Wiener.

Author's summary.

TOPOLOGY

Inagaki, Takeshi, and Sugawara, Masahiro. Compactification of topological spaces. Math. J. Okayama Univ. **2**, 85–97 (1952).

This paper contains a generalization of and unified approach to the Čech-Wallman compactification theorems [Čech, Ann. of Math. (2) **38**, 823–844 (1937); Wallman, Proc. Nat. Acad. Sci. U. S. A. **23**, 164–165 (1937)]. In an earlier note the first author made a T-space R^* of the set of all ultrafilters in a T-space R so that (A) R^* is compact and has a dense subset R' homeomorphic to R. This paper shows first that (B) each continuous bounded real-valued function on R' can be extended to R^*.

If $F_i\in R^*$, write $F_1\sim F_2$ if $\bar{F}_1$, the set of elements of F_1 which are closed subsets of R, coincides with $\bar{F}_2$. Let R^0 be the space of these equivalence classes, and let $\beta(R)$ be the subspace of R^0 containing those classes $[F]$ for which $\bar{F}$ is a maximal dual ideal in the lattice of closed subsets of R. Write $F_1\approx F_2$ if they cannot be separated by real-valued continuous functions on R^*, and let $\gamma(R)$ be the space of these equivalence classes. Theorem 1 (2) [3]. If R is a T_0 (T_1) [completely regular] space, then R^0 ($\beta(R)$) [$\gamma(R)$] is a space of the same kind and has properties (A) and (B).

M. M. Day (Urbana, Ill.).

Terasaka, Hidetaka. On Cartesian product of compact spaces. Osaka Math. J. **4**, 11–15 (1952).

Using, essentially, the β-compactification of a countable discrete space, the author gives an example of a countably compact completely regular space R such that $R\times R$ is not countably compact. The author points out that the question of the existence of such a space was answered by him in 1947 [in an unavailable paper in Japanese]; he remarks, too, that a similar example has been given by J. Novák [announced in Časopis Pěst. Mat. Fys. **74**, 238–239 (1950)].

M. Katětov (Prague).

Stone, A. H. On infinitely multicoherent spaces. Quart. J. Math., Oxford Ser. (2) **3**, 298–306 (1952).

Given a connected topological space S of infinite degree of multicoherence $r(S)$, a study is made of conditions under which $r(S)$ is actually attained, i.e., when does S admit a decomposition into two closed connected sets whose intersection has infinitely many components? If S is a one-dimensional Peano space, it is shown that $r(S)$ is always attained in this sense. Also a general theorem on Cartesian product spaces is established which provides a procedure of construeting Peano spaces of infinite dimensionality in which $r(S)$ is not attained. This latter procedure is modified so as to exhibit a Peano space of dimension 2 in which also $r(S)$ is not attained.

G. T. Whyburn.

Sitnikov, K. A. On continuous mappings of open sets of a Euclidean space. Mat. Sbornik N.S. **31(73)**, 439–458 (1952). (Russian)

Let Γ be an open subset of R^n with boundary $\dot{\Gamma}$. Let φ be a continuous mapping of Γ into R^n, and denote by F_x the set $\varphi^{-1}(\varphi(x))$, for all $x\in\Gamma$. Let o be a certain point of Γ. If diam $F_x/\rho(x,o)$ goes to zero as x approaches $\dot{\Gamma}$, then $\varphi\Gamma$ is open and $\varphi\Gamma$ is homologically equivalent to Γ. This is a far-reaching generalization of Brouwer's theorem on the invariance of domain (φ a homeomorphism) and of a theorem of Borsuk [Fund. Math. **21**, 236–243 (1933)].

E. Hewitt (Seattle, Wash.).

Bourgin, D. G. Sets of visibility. Portugaliae Math. **11**, 137–140 (1952).

Suppose E is a linear topological space, S a subset of E, and M the smallest linear manifold containing S. For $x\in S$, let $v(x)=\{y\,|\,[x,y]\subset S\}$, where $[x,y]$ is the closed segment joining x to y, and let $w(x)$ be the set of all $y\in v(x)$ such that no subsegment of $[x,y]$ is contained in the boundary of S relative to M. Let $V(S)=\{x\,|\,v(x)=S\}$ and $W(S)=\{x\,|\,w(x)=S\}$.

The author shows that if K is a circular disk, then K is properly contained in a closed plane set S such that $V(S)=W(S)=K$. *V. L. Klee, Jr.*

White, Paul A. Extensions of the Jordan-Brouwer separation theorem and its converse. Proc. Amer. Math. Soc. 3, 488–498 (1952).

In the reviewer's Colloquium book [Amer. Math. Soc. Colloq. Publ., vol. 32, New York, 1949; these Rev. **10**, 614], certain results related to extensions of the classical Jordan-Brouwer separation theorem contained the hypothesis that the $(n-1)$-dimensional Betti number of the imbedding manifold be zero. The present article aims in large part at removing this condition and making further extensions. On the basis of a definition of n-dimensional generalized manifold (n-gm) with boundary, it is shown that if, in an orientable n-gcm S (always perfectly normal and connected), K is an $(n-1)$-gcm which is the common boundary of disjoint open sets A and B whose union is $S-K$, then K is orientable and both $K\cup A$, $K\cup B$ are orientable n-gms with boundary K. If the hypothesis is further weakened so that $S-K$ is merely assumed disconnected, then the closure of each component of $S-K$ is an orientable n-gm with boundary consisting of some of the components of K. However, the latter result depends on a theorem, viz. Theorem 2, whose proof is insufficient (the sets $A_1\cap S'$, $A_2\cap S'$ are not shown to be both non-empty); the introduction of such a condition as that K is locally orientable seems necessary to validate the theorem and those results of the paper that subsequently depend on it. The concluding results of the paper are concerned with open subsets of a manifold that satisfy certain uniform local connectedness conditions. Thus, with S as before, if A is an open r-ulc $(r=0, 1, \cdots, n-1)$ subset of S, with $(n-1)$-dimensional boundary K, then $\bar{A}$ is an n-gm with boundary K. *R. L. Wilder.*

Georgiev, G. Sur certains automorphismes à points fixes des surfaces fermées orientables. Acta Math. Acad. Sci. Hungar. 3, 71–72 (1952). (Russian summary)

An elementary proof of the following theorem: If f is a homeomorphism of an orientable surface of positive genus onto itself, if C is a simple closed curve on the surface which is homotopic to a point, and if $f(C)=C$, then f has a fixed point. *E. G. Begle* (New Haven, Conn.).

Fan, Ky. A generalization of Tucker's combinatorial lemma with topological applications. Ann. of Math. (2) 56, 431–437 (1952).

The author generalizes a combinatorial lemma due to Tucker [Proc. First Canadian Math. Congress, Montreal, 1945, Univ. of Toronto Press, 1946, pp. 285–309; these Rev. **8**, 525], and uses his result to prove two theorems, one a generalization of the Borsuk-Ulam theorem, and the second a generalization of the Lusternik-Schnirelmann theorem. We quote his second theorem: Let n, m be two arbitrary positive integers. If m closed subsets $F_1, F_2, \cdots, F_m$ of the n-sphere S^n cover S^n and if no one of them contains a pair of antipodal points, then there exist $n+2$ indices $k_1, k_2, \cdots, k_{n+2}$ with $1\leqq k_1<k_2<\cdots<k_{n+2}\leqq m$ such that $F_{k_1}\cap -F_{k_2}\cap F_{k_3}\cap\cdots\cap(-1)^{n+1}F_{k_{n+2}}\neq 0$, where $-F_i$ denotes the antipodal set of F_i. In particular, m is necessarily $\geqq n+2$. The partial result that $m\geqq n+2$ is, of course, the Lusternik-Schnirelmann theorem. *E. G. Begle.*

Puppe, S. D. Minkowskische Einheiten und Verschlingungsinvarianten von Knoten. Math. Z. 56, 33–48 (1952).

Let k be a simple closed polygon in spherical 3-space S. Associated with each regular plane projection of k is a quadratic form Q. It is known [cf. Reidemeister, Knotentheorie, Springer, Berlin, 1932] that the Minkowski units $C_p(Q)$, p ranging over the odd primes, of Q are independent of the regular projection chosen and are unaltered if k is subjected to any combinatorial isotopy; thus the units $C_p(Q)$ are invariants of the combinatorial isotopy type of k. Also associated with k is the double covering S_2 of S branched over k, and the linking invariants $v_{p^s}(G)$ of the 1-dimensional torsion group G of S_2, p^s ranging over certain powers of the torsion coefficients of G. It is known [Seifert, Abh. Math. Sem. Univ. Hamburg 11, 84–101 (1935)] that the $C_p(Q)$ do not determine the $v_{p^s}(G)$; it was an open question whether or not the $v_{p^s}(G)$ determine the $C_p(Q)$. This question is now answered in the affirmative; it is shown that $C_p=((-1)^{\lambda/2}/p)\prod_{s\equiv 1 (\mathrm{mod}\ 2)}v_{p^s}$, where $({}^*/p)$ denotes the Legendre symbol and λ is the number of torsion coefficients that contain the prime p to an odd power. This important theorem is the main result of the paper.

The author claims that the v_{p^s}, and hence, by the above theorem, the C_p are invariants of the combinatorial isotopy type of k. It seems to the reviewer that this does not follow immediately from the argument on p. 43 unless recourse is had to a theorem announced by the reviewer [Proc. Internat. Congress Math., Cambridge, Mass., 1950, v. 2, Amer. Math. Soc., Providence, R. I., 1952, pp. 453–457, esp. p. 455; these Rev. **13**, 966] which states in particular that S_2 is a topological invariant of the knot type of k. If this is done, there results the stronger statement that the v_{p^s} and the C_p are invariants of the (topological) knot type of k. The author uses a definition of the Minkowski units in terms of Gauss sums. This definition, which seems to be new, appears to have certain advantages over the usual definition (the two definitions are, of course, proved to be equivalent).

Reviewer's note: The principal theorem of this paper was obtained independently by the late R. H. Kyle [thesis, Princeton, 1951]. Publication of this somewhat more comprehensive study (the prime 2 is considered) has been delayed by the editing of his posthumous papers.

R. H. Fox (Princeton, N. J.).

Borel, Armand. Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts. Ann. of Math. (2) 57, 115–207 (1953).

This is a detailed account of results previously announced by the author [C. R. Acad. Sci. Paris **231**, 943–945 (1950); **232**, 2392–2394 (1951); **233**, 569–571 (1951); these Rev. **12**, 435; **13**, 56, 319]. The object of the study is the homology relations of a principal fiber bundle with a compact Lie structural group and the applications of these results to the determination of the homology groups of homogeneous spaces. For differentiable fiberings and relative to real coefficients the problem has been studied by Cartan-Chevalley-Koszul-Weil and others by algebraic methods. The main contribution of the author lies in the results concerning more general coefficients, in particular, coefficients mod p. His major tools are Leray's spectral cohomology theory and the notion of transgression.

The paper is divided into seven chapters. Chapter I contains the preliminaries, with an introduction to the spectral cohomology theory, so as to make the paper as self-contained as possible. Four definitions of transgression are given, of which the following is perhaps the one best known:

Let E, B, F, p be respectively the fiber space, the base space, the fiber, and the projection of a fiber bundle, and let M be a coefficient field. An element $h \in H^s(F_b, M)$, $F_b = p^{-1}(b)$, $b \in B$, is called transgressive, if there is a cochain $c \in C^s(E, M)$, whose inverse image under the inclusion mapping $F_b \subset E$ is a cocycle belonging to h and whose coboundary dc is a cocycle in B. The cohomology class of dc is determined up to an element of a submodule $K^{s+1}(B, M)$ of $H^{s+1}(B, M)$. The transgression is the homomorphism of the submodule of transgressive elements of $H^s(F, M)$ into $H^{s+1}(B, M)/K^{s+1}(B, M)$.

Chapter II gives an algebraic generalization of Hopf's theorem on the cohomology of Γ-manifolds, which the author calls, quite reasonably, Hopf manifolds. Hopf's theorem is essentially algebraic. It is concerned with the structure of a graded algebra H over a field K_p (to be called a Hopf algebra), with the unit element 1, whose multiplication of homogeneous elements is anticommutative, such that there is a graded multiplicative homomorphism $f: H \rightarrow H \otimes H$ and there are two automorphisms ρ and σ of H, satisfying the condition

$$f(h) = \rho(h) \otimes 1 + 1 \otimes \sigma(h) + \sum_{i=1}^{r} x_i \otimes y_i, \quad 0 < \deg (x_i) < \deg h,$$

for any homogeneous element $h \in H$. It is proved that, when K_p is a perfect field and when a further assumption on the existence of generators is satisfied, a Hopf algebra is isomorphic to a skew tensor product of Hopf algebras, each with one generator. Topological consequences of this theorem are drawn, among which is the theorem that if X is a connected compact Hopf manifold without p-torsion, its cohomology algebra with coefficients mod p is the exterior algebra of a vector space spanned by elements of odd degree. As applications of these results and for future use, there is given in Chapter III a complete determination of the additive homology structure of real, complex, and quaternion Stiefel manifolds.

Chapter IV contains what the author considers to be the main result of this paper. It is a purely algebraic theorem on spectral sequences of graded algebras, whose topological analogue is the spectral cohomology structure of a universal principal fiber bundle. Topological applications of this theorem are made in Chapter V. The problem at issue is to determine, for a principal fiber bundle with compact Lie structural group G, the submodule of its transgressive elements in $H^s(G, M)$, together with its image in $H^{s+1}(B, M)/K^{s+1}(B, M)$. The problem is easily reduced to one on universal bundles. Corresponding to the dimension n, a universal space $E(n, G)$ is defined as a principal fiber bundle, with fiber G, which is compact, connected, locally connected and whose cohomology groups (in the sense of Alexander-Spanier, with arbitrary coefficients) are trivial up to the dimension n. The base space $B(n, G) = B_G$ of this bundle is called a classifying space for G and for the dimension n. Universal spaces are easily seen to exist. It is proved that the cohomology algebras of two classifying spaces for G and for dimensions $\geqq n$ are canonically isomorphic up to dimension n. An element $h \in H(G, M)$ is called universally transgressive, if there is a universal bundle for arbitrarily large dimensions in which it is transgressive. If $H(G, K_p)$, where K_p is the field mod p, is an exterior algebra of a vector space with a base formed by elements of odd degrees, then $H(G, K_p)$ has a system of generators, which are universally transgressive. Their images in $H(B_G, K_p)$ generate a polynomial algebra, which is the cohomology algebra of B_G up to the dimension n. The universally transgressive elements of $H(G, K_p)$ are intimately related to the primitive elements. An element $x \in H(G, K_p)$ is called primitive if its image $f^*(x)$ under the dual homomorphism of the mapping $f: G \times G \rightarrow G$ defined by group multiplication is given by $f^*(x) = 1 \otimes x + x \otimes 1$. For real coefficients the universally transgressive elements are known to be identical with the primitive elements, and this theorem is generalized here to K_p. This generalization is useful in the definition of a natural homomorphism $\rho^*(U, G)$: $H(B_G, M) \rightarrow H(B_U, M)$, where U is a closed subgroup of G; ρ^* plays an important rôle in the study of homogeneous spaces. Chapter V concludes with a discussion of the classifying spaces when G is the rotation group. As is known, the latter can be taken to be Grassmann manifolds.

Applications of the general theorems developed above are made in Chapter VI for the case of real coefficients and in Chapter VII for coefficients mod p, the latter being less complete than the former. For real coefficients the author derives, among other results, theorems of Koszul, H. Cartan, and a formula of Hirsch concerned with the Betti numbers of a homogeneous space whose group of transformations and group of isotropy have the same rank. It is also shown how the Betti numbers of a connected compact Lie group can be read from the diagram in the universal covering space of a maximal toroid, as acted on by the Weyl group. Results in Chapter VII are mainly concerned with homogeneous spaces G/U, where G and U have the same rank. Several particular cases are worked out in detail. As a side result it is proved that G/T, where G is semi-simple and T its maximal toroid, has a complex structure invariant under the homeomorphisms of G. *S. Chern* (Chicago, Ill.).

***Thom, R. Une théorie intrinsèque des puissances de Steenrod.** Colloque de Topologie de Strasbourg, 1951, no. VII, 13 pp. La Bibliothèque Nationale et Universitaire de Strasbourg, 1952.

***Wu, Wen-tsün. Sur les puissances de Steenrod.** Colloque de Topologie de Strasbourg, 1951, no. IX, 9 pp. La Bibliothèque Nationale et Universitaire de Strasbourg, 1952.

These two papers may be conveniently reviewed together, since the main results apparently depend on the joint work of both authors. The two principle theorems are axiomatic characterizations of the reduced pth powers of Steenrod [cf. the references below] for the case where p is prime.

The case where $p=2$ is much simpler than the case $p>2$, and may be considered separately. Suppose there is defined for each pair (E, F) consisting of a topological space E and a subspace F and for each integer $i \geqq 0$ a homomorphism $D^i: H^r(E, F) \rightarrow H^{r+i}(E, F)$ of the relative cohomology groups with mod 2 coefficients which satisfies the following three conditions. (1) D^0 is the identity; if $x \in H^r(E, F)$, then $D^i(x) = 0$ for $i > r$, and $D^r(x) = x \smile x$ (cup product). (2) For a cup product, we have the formula of H. Cartan: $D^p(u \smile v) = \sum_i D^i(u) \smile D^{p-i}(v)$. (3) If $\delta: H^r(F) \rightarrow H^{r+1}(E, F)$ is the coboundary homomorphism, then δ commutes with D^i. The first theorem then asserts that D^i must be the Steenrod square, Sq^i [for the definition, cf. Ann. of Math. (2) **48**, 290–320 (1947); these Rev. **9**, 154]. It is interesting to note that it is not necessary to assume the "naturality condition" (i.e., if $f: E \rightarrow F$ is a continuous map, then the induced homomorphism $f^*: H^n(F) \rightarrow H^n(E)$ commutes with D^i) as an axiom; it follows from the above conditions.

The statement of the axioms for the case $p>2$ reads as follows. Suppose there are defined for each integer j and each pair (E, F) a set of homomorphisms D^j:

$H^r(E, F)\to H^{r+j}(E, F)$, where the cohomology groups have mod p coefficients, subject to the following conditions: (1) If $x \in H^r(E, F)$, then $D^j(x)=0$ in case j is negative, or j is odd, or $j>r(p-1)$; D^0x is a non-zero integral multiple of x; if $j=r(p-1)$, then $D^j(x)=x^p$. (2) For cup products, we have the formula of H. Cartan, $D^q(u\smile v)=\sum_i D^i(u)\smile D^{q-i}(v)$, as above. (3) If δ is the coboundary operator, then $D^j\delta=m\delta D^j$, where m is a non-zero integer. (4) The naturality condition holds, i.e., D^j commutes with the homomorphism induced by a continuous map. The second theorem then asserts that for any r-dimensional cohomology class x and any even integer j, $D^j(x)$ differs only by a trivial factor from the Steenrod cyclic reduced pth power, $\mathcal{P}_i^p(x)$, where $i=r(p-1)-j$ [for the definition of the operator $\mathcal{P}_i^p$, cf. Ann. of Math. (2) 56, 47–67 (1952); these Rev. 13, 966]. It is known that the operators $\mathcal{P}_i^p$ for $(p-1)r-i$ odd may be deduced from those for $(p-1)r-i$ even.

The proofs of these theorems depend on the development of some rather elaborate machinery. The main tool used is P. A. Smith's theory of periodic transformations of topological spaces [cf. appendix B of S. Lefschetz, Algebraic Topology, American Mathematical Society, New York, 1942; these Rev. 4, 84] applied to the following case: Let E be a topological space, E^p the cartesian product of p copies of E, and $T: E^p\to E^p$ a homeomorphism defined by permuting the coordinates of a point of E^p cyclically. Then T is of period p, the fixed point set of T is the "diagonal" of E^p, and the Smith-Richardson theory may be applied to determine the cohomology groups of the quotient space (the p-fold cyclic product of E). Thom gives a neat reformulation of this theory in modern terminology which is of interest in itself, and Wu proves two key theorems. One of them shows that certain of the operators of the Smith-Richardson theory may be expressed in terms of cup products.

As a consequence of the axiomatic characterization of the reduced pth powers ($p>2$), Thom proves that $D^j(x)=0$ unless j is divisible by $2(p-1)$.

The paper of Wu concludes with a discussion of the reduced pth powers in the Grassmann manifolds, and the reduced powers of the Stiefel-Whitney, Chern, and Pontrjagin classes in a sphere bundle.

The exposition in these papers is rather condensed, which does not make for easy reading. To make matters worse, the typography is so bad that most of the formulas cannot be read; they can only be conjectured. It is to be hoped that the authors will publish a complete readable exposition in the near future. *W. S. Massey* (Providence, R. I.).

★Thom, R. Quelques propriétés des variétés-bords. Colloque de Topologie de Strasbourg, 1951, no. V, 10 pp. La Bibliothèque Nationale et Universitaire de Strasbourg, 1952.

An important problem in topology is to determine necessary and sufficient conditions in order that an n-manifold V should be the boundary of some $(n+1)$-manifold, M, i.e., that the pair (M, V) should be a manifold with boundary. In this paper, the author systematically studies the problem of determining necessary conditions under the assumption that V is compact. He first proves a generalized duality theorem, under the additional assumption that M is also compact: Then the homology group $H_p(M)$ is isomorphic to the relative cohomology group, $H^{n+1-p}(M \bmod V)$, for the complementary dimension. Similarly, $H^p(M)$ is isomorphic to $H_{n+1-p}(M \bmod V)$. This theorem is established without any hypotheses of triangulability, differentiability, or regularity of the boundary.

Let A^p denote the image of the injection $H^p(M)\to H^p(V)$, and K_q the kernel of the injection $H_q(V)\to H_q(M)$. The author's second theorem states that A^p and K_{n-p} are isomorphic, without any assumption of compactness on M. This theorem has many important consequences. Some of the most important of these are the following: (1) Suppose that M and V are orientable, and we use a field for coefficients. Then the cup product defines a pairing of $H^p(V)$ and $H^{n-p}(V)$ to $H^n(V)$. Under this pairing, A^p and A^{n-p} are the annihilators of each other. If b_p denotes the pth Betti number of V, and r_p denotes the rank of the vector space A^p, then $b_p=b_{n-p}=r_p+r_{n-p}$. Hence the Euler characteristic of V is even. (2) Suppose $n=4k$, and $p=2k$. Then the cup product pairing of $H^p(V)$ with $H^p(V)$ to $H^n(V)$ defines a quadratic form Φ on $H^p(V)$, and A^p is a null-space of Φ. If one uses the real field for coefficients, and chooses a basis for $H^p(V)$ so that Φ reduces to the diagonal form, then Φ must contain an equal number of positive and negative squares. (3) The pth Stiefel-Whitney class, W^p, is an element of A^p (coefficients mod 2).

The author defines two oriented n-manifolds V and V' (which need not be connected) to be cobounding if there exists an $(n+1)$-manifold with boundary M having $V\cup V'$ for boundary, and such that if m denotes the fundamental cycle of M mod $(V\cup V')$, then the boundary of m is $v-v'$, where v and v' are the fundamental cycles of V and V'. The manifolds V and V' are said to be simply cobounding if one ignores the orientability conditions in the above definition. The relation "V and V' are cobounding" is an equivalence relation, and the set of all equivalence classes becomes a ring if we define addition in the obvious way, and multiplication to be the formation of the cartesian product of two manifolds. The author determines all equivalence classes for some of the lower dimensions.

The last part of the paper is devoted to applying the preceding results to the study of real algebraic varieties. Among other things, it is proved that if an n-manifold V is the locus of zeros in Euclidean n-space of an ideal generated by k polynomials, then V is a bounding manifold. If an orientable n-manifold V can be imbedded in Euclidean $(n+2)$-space, then V is a bounding manifold. *W. S. Massey* (Providence, R. I.).

Deheuvels, René. Anneau différentiel à filtration réelle. C. R. Acad. Sci. Paris 235, 778–780 (1952).

Deheuvels, René. L'anneau local d'un anneau différentiel à filtration réelle; ses relations avec l'anneau d'homologie. C. R. Acad. Sci. Paris 235, 858–860 (1952).

Deheuvels, René. Invariants topologiques d'une fonctionnelle semi-continue inférieurement sur un espace localement compact. C. R. Acad. Sci. Paris 235, 929–930 (1952).

Deheuvels, René. Calcul des variations et cohomologie singulière. C. R. Acad. Sci. Paris 235, 1270–1272 (1952).

This series of four notes deal with a differential ring with a real filtration and its application to calculus of variation. All the results are stated without proof. In the first note the author lays down the preliminary definitions. Critical rings, critical values, and derived rings are defined. The second note is devoted to the relation between the homology ring and the local ring of a differential ring with a real filtration. The theory of rank and span of M. Morse [Ann. of Math. (2) 41, 419–454 (1940); these Rev. 1, 320] and a

theorem of Froloff and Elsholz [Mat. Sbornik **42**, 637–643 (1935)] are generalized. All the definitions and statements in the first two notes are purely algebraic. The third note exposes a multiplicative cohomology theory of critical values of a lower semi-continuous functional on a locally compact space. The essential idea is taken from a memoire of J. Leray [J. Math. Pures Appl. (9) **29**, 1–139 (1950); these Rev. **12**, 272]. In the last note the notion of accessible filtration is introduced and it is a generalization of the φ-accessibility hypothesis of M. Morse [Functional topology and abstract variational theory, Gauthier-Villars, Paris, 1939]. Then the author states that the real filtration of the ring of singular cochains on an arbitrary space X, induced by a numerical function on X, is accessible.

C.-T. Yang (Urbana, Ill.).

Zykov, A. A. On some properties of linear complexes. Amer. Math. Soc. Translation no. 79, 33 pp. (1952).

Translated from Mat. Sbornik N.S. **24**(66), 163–188 (1949); these Rev. **11**, 733.

GEOMETRY

Guitel, Geneviève. Principes de classification dans l'étude des trièdres et des tétraèdres. C. R. Acad. Sci. Paris **235**, 1274–1276 (1952).

This is a very condensed account of what appears to be the most complete study ever undertaken of the dihedral angle, the tetrahedron, and the tetrahedral angle, as to their face and dihedral angles.

The author has proved that if a trihedron (1) has only one obtuse face angle, it has one and only one obtuse dihedral angle; (2) all three of its dihedral angles are obtuse, the same holds for its face angles; (3) the face angles a, b, c satisfy the relations $a>\pi/2$, $c=\pi-a$, $\cot(b/2)=\sin a$, the trihedron is congruent to its supplementary trihedron. Twenty-seven realisable species of tetrahedrons are considered. One species has only acute dihedral angles, six species have one obtuse dihedron, eleven species have two opposite obtuse dihedrons, etc.

In addition to the convex tetrahedral angle, the author considers two other kinds: the tetracant and the inversed tetracant. A tetracant is defined as a tetrahedral angle such that the half-line opposite to any one of the edges of the angle is interior to the trihedron determined by the three remaining edges. An inversed tetracant is obtained from a tetracant by replacing one of its edges by the opposite half-line. All three kinds of tetrahedral angles are obtained by taking for their edges the perpendiculars dropped from a point in space upon the faces of a tetrahedron. A tretracant, an inversed tetracant, or a convex tetrahedral angle are obtained according as the point is taken (1) inside the tetrahedron, (2) in a trunc, or in the trihedral angle symmetric to a trihedral angle of the tetrahedron, (3) in a roof of the tetrahedron. The twenty-seven species of tetrahedrons give rise to various species of tetrahedral angles.

N. A. Court (Norman, Okla.).

Court, N. A. Orthological triangles. Math. Student **20**, 51–57 (1952).

Court, Nathan Altshiller. Sur les tétraèdres orthologiques. Mathesis **61**, 249–256 (1952).

Thébault, Victor. Sur le point de Monge d'un tétraèdre. Mathesis **61**, 281–287 (1952).

Guillotin, R. Sur une cubique et deux familles de triangles associés. Mathesis **61**, 269–277 (1952).

Lowry, H. V. Polygons inscribed in polygons. Math. Gaz. **36**, 256–262 (1952).

Given a polygon $A=A_1A_2\cdots A_n$, the author shows that the problem of inscribing in A a polygon $P=P_{01}P_{12}\cdots P_{n1}$ of given orientation and given shape has always one and only one solution, that is, there is a unique position for the point P_{01}, on the side A_{n1} of A, for which P_{n1} coincides with P_{01}. The conditions imposed on P remaining the same, the point P_{01} is taken so that P_{n1} does not coincide with P_{01}, and the polygon $Q=Q_{01}Q_{12}\cdots Q_{n1}$ is constructed with $Q_{01}=P_{n1}$ and the sides of Q parallel to the respective sides of P. The author investigates the possibility of $Q_{n1}=P_{01}$ and arrives at the conclusion that for certain directions of the sides of P this condition of closure will be realized for every point P_{01} of the side A_nA_1 of A. The case when A is a triangle is given special attention. Ten interesting figures illustrate the discussion. *N. A. Court* (Norman, Okla.).

Bowman, F. Cyclic pentagons. Math. Gaz. **36**, 244–250 (1952).

Möbius considered the problem of finding all the cyclic configurations of a polygon of which the lengths of the sides are given [J. Reine Angew. Math. **3**, 5–34 (1828)]. In particular, he indicated that an equation of the seventh degree expressed all the solutions for the pentagon. The present paper derives this equation, the length of a diagonal in terms of the given sides, in explicit form. Since each solution is one of twelve possible permutations, there may be as many as 84 cyclic configurations. At least one of the seven basic solutions, the convex case, is real. The equation is obtained from 30 equations expressing the repeated application of Ptolemy's theorem: the product of the diagonals of a quadrilateral is equal to the sum of the products of pairs of opposite sides. *M. Goldberg* (Washington, D. C.).

Locher-Ernst, L. Wie viele regelmässige Polyeder gibt es? Arch. Math. **3**, 193–197 (1952).

In this note the author regards a polyhedron as being "regular" when it has twice as many symmetry operations as edges. The number of such polyhedra (including compounds) is found to be twelve [cf. H. Jensen, Mat. Tidsskr. A. **1945**, 72–77; these Rev. **8**, 83]. *H. S. M. Coxeter.*

Cundy, H. Martyn. "Deltahedra." Math. Gaz. **36**, 263–266 (1952).

The name "deltahedron" is used for a polyhedron whose faces are equilateral triangles. Convex deltahedra were completely enumerated by Freudenthal and van der Waerden [Simon Stevin **25**, 115–121 (1947); these Rev. **9**, 99]. The author describes 17 non-convex deltahedra, each having two types of vertex while the faces do not cross one another. It is of some historical interest to record that six of these (nos. 1, 7, 6, 4, 8, 11) are the solids considered in chapters XLVIII to LIII of Luca Pacioli, Divina proportione [Venice, 1509; Editorial Losada, Buenos Aires, 1946], with illustrations by Leonardo da Vinci.

H. S. M. Coxeter (Toronto, Ont.).

Pipping, Nils. Drei geometrische Miniaturen. Acta Acad. Aboensis **18**, no. 10, 8 pp. (1952).

The author describes a construction due to A. Heiseler [Acta Acad. Aboensis **8**, no. 8 (1935)] involving four circles and two straight lines, for a segment of length $(\sqrt{141}-\sqrt{6})/3$, which differs from π by less than 0.000025. He then gives a new construction, involving seven circles (and no lines), for the "golden section" of a given segment. Finally, he simplifies the classical use of the cissoid for duplicating the cube. *H. S. M. Coxeter* (Toronto, Ont.).

Coxeter, H. S. M. Interlocked rings of spheres. Scripta Math. **18**, 113–121 (1952).

The envelope of a varying sphere that touches three fixed spheres is a Dupin cyclide, which by a suitable inversion can be transformed into a torus. Hence the cyclide may be described as an envelope of spheres in two ways: each of the two families consists of spheres touching any three members of the other. Any sphere touches two of the same family. We consider the case that the cyclide is of such a shape that there is a closed ring of n spheres, each touching the next. The property is poristic: if there is one ring, then every sphere of the same family belongs to such a ring. If the ring goes round the cyclide d times, it is called a p-ring, where $p=n/d$. The author proves a theorem of Steiner: if one family has p-rings, then the other has q-rings, where $1/p+1/q=1/2$. The proof is based on a consideration of certain four-dimensional polytopes [Coxeter, Regular polytopes, Methuen, London, 1948; these Rev. **10**, 261] and the stereographic projection of the in-spheres of its cells on three-dimensional space. The author proves that any "interlocked" p-ring and q-ring can be derived in this manner and he obtains many interesting properties of the configuration. Special cases are $p=3$, $q=6$ (Soddy's hexlet) and $p=q=4$. *O. Bottema* (Delft.).

Aiyer, K. Rangaswami. On the structure of Joachimstal's circles of a conic. Bull. Calcutta Math. Soc. **43**, 139–142 (1951).

Hameed, Asghar. A quadric associated with two points. Pakistan J. Sci. Res. **3**, 48–51 (1951).

Tummers, J. H. Une certaine transformation. Simon Stevin **29** (1951/52), 77–82 (1952).

La transformation $X:Y:Z=x(y^2-z^2):y(z^2-x^2):z(x^2-y^2)$ fait correspondre à toute droite une cubique; l'A. étudie les points particuliers de cette cubique et les sections par des droites passant en certains de ces points. Il indique ainsi des relations remarquables entre des cubiques particulières, celles de MacKay, de Darboux, des dix-sept points et des points connus de la géométrie du triangle comme ceux de Longchamp et de Lemoine. *B. d'Orgeval* (Alger).

Lasley, J. W., Jr. On the projective-metric definition of distances. J. Elisha Mitchell Sci. Soc. **67**, 96–98 (1951).

The author has rediscovered the transition from non-Euclidean to Euclidean analytic geometry, as described by R. Bonola [Non-Euclidean geometry, Open Court, Chicago, 1912, pp. 162–163]. *H. S. M. Coxeter* (Toronto, Ont.).

Herrmann, Horst. Matrizen als projektive Figuren. Jber. Deutsch. Math. Verein. **56**, Abt. 1, 6–20 (1952).

The author represents a simplex in projective n-space by a square matrix whose columns contain the coordinates of the vertices; e.g., the unit matrix represents the simplex of reference. When n is odd, a null polarity transforms the simplex of reference into a simplex represented by a skew-symmetric matrix, and we have the generalization of a Möbius pair of tetrahedra. Similarly (for any n) a symmetric polarity transforms the simplex of reference into one represented by a symmetric matrix, and we have a pair of simplexes whose corresponding vertices are joined by $n+1$ lines which are "associated" in the sense that every $(n-2)$-space meeting n of them meets the remaining one too. [For the case $n=3$, see H. F. Baker, Principles of geometry, vol. 3, Cambridge, 1923, p. 41, Ex. 7.] The author has communicated to the reviewer his regrets that, in §3, he repeated the mistake in §158 of his Übungen zur projektiven Geometrie [Birkhäuser, Basel, 1952; these Rev. **14**, 308].

H. S. M. Coxeter (Toronto, Ont.).

Kelly, L. M. The geometry of normed lattices. Duke Math. J. **19**, 661–669 (1952).

Notations and definitions for a metric space M—b-cover $B(p, q)$: set of the points lying between p and q including p and q. The subset S of M is "completely convex" if $((p \in S)\&(q \in S))\rightarrow(B(p, q)\subseteq S)$. S is "of constant width" if for each point p of S there is a point p' of S with distance pp' = diameter of S. If such a diametral point is unique for every p, the set S is "round". A "tripod" is a system of four (distinct) points p, q, r, s such that s is between each pair of the remaining points; p, q, r are the "feet" and s the "vertex" of the tripod. First Part—Glivenko [Amer. J. Math. **58**, 799–828 (1936)] and Smiley and Transue [Bull. Amer. Math. Soc. **49**, 280–287 (1943); these Rev. **4**, 248] have formulated conditions for a metric space M to be the associated metric space $D(L)$ of a normed lattice L with a first element. A set of three conditions (a), (b), (c) is given which are applicable to any normed lattice: (a) Each three points of M are contained in a completely convex b-cover (special transitivity of the metric betweenness); (b) each completely convex b-cover C is almost ordered relative to at least one point $o(C)$ of that cover; (c) $o(C)$ can be chosen so that if a and b are in C' and C'', then (b lies between a and $o(C')$)$\rightarrow$(b lies between a and $o(C'')$). Second Part—The author tackles the problem of reconstructing L from $M\equiv D(L)$. Propositions: (L is complemented)$\equiv$($D(L)$ is of constant width). (L is a Boolean algebra)$\equiv$($D(L)$ is round). (L is distributive)$\equiv$(Each three points of $D(L)$ are the feet of a unique tripod). (L is a chain)$\equiv$($D(L)$ satisfies the Ptolemaic inequality). (A subset J of L with a 0 is an ideal)$\equiv$(J is completely convex and contains 0).

C. Pauc (Nantes).

Seidel, J. Distance-geometric development of two-dimensional euclidean, hyperbolical and spherical geometry. II. Simon Stevin **29** (1951/52), 65–76 (1952).

In part I [Simon Stevin **29**, 32–50 (1952); these Rev. **14**, 75] the author gave a distance-geometric development of euclidean plane geometry, based on properties of the Cayley-Menger determinant D. This program is continued in the present work, where the foundations of two-dimensional spherical and hyperbolic geometries are related to properties of the analogous determinants in the cosines or hyperbolic cosines of the mutual distances of finite subsets.

L. M. Blumenthal (Columbia, Mo.).

Convex Domains, Extremal Problems, Integral Geometry

Fenchel, W. A remark on convex sets and polarity. Comm. Sém. Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.] Tome Supplémentaire, 82–89 (1952).

The author calls a set C in A^n (n-dimensional affine space) "evenly convex" provided it is the intersection of a family of open half-spaces. Some characterizations of such sets are given, and in particular a set C in A^n is shown to be evenly convex if and only if C is connected and each point not in C lies on a plane which has no point in common with C. In the projective space Π^n, this property is taken as the definition of even convexity. A type of polarity is defined in Π^n and is shown to be involutary when applied to evenly convex sets. For A^n this yields the following result (classical if $<$ is replaced by $\leqq$ and "evenly convex" by "closed and convex"): For each set C in A^n which contains the origin, let C^* be the set of all $u \in A^n$ such that $x\cdot u<1$ whenever $x \in C$ ($\cdot$ denoting inner product). Then C is evenly convex and $C^{**}=C$ if and only if C is evenly convex.

V. L. Klee, Jr. (Charlottesville, Va.).

Green, John W. On the chords of a convex curve. II. Portugaliae Math. 11, 51–55 (1952).

[For part I see Portugaliae Math. 10, 121–123 (1951); these Rev. 13, 571]. Let C be a closed convex plane curve of width w and diameter d. A chord of C is said to be right-subtended if its ends meet a pair of orthogonal supporting lines of C, to be right-subtended by a rectangle R circumscribing C if its ends meet a pair of adjacent sides of R. It is shown that for every C and circumscribing R there exist four chords of C right-subtended by R whose lengths l_k satisfy:

$$\frac{1}{2}<\frac{l_1}{d},\quad \frac{l_2}{d}\leqq\frac{1}{\sqrt{2}},\quad \frac{1}{\sqrt{2}}\leqq\frac{l_3}{w},\quad \frac{l_4}{w}<\infty,$$

each of these bounds being optimum. Moreover, for every C there exist four chords whose lengths l_k satisfy:

$$\frac{\sqrt{7}-1}{3}<\frac{l_1}{d},\quad \frac{l_2}{d}\leqq\frac{1}{\sqrt{2}},\quad \frac{1}{\sqrt{2}}\leqq\frac{l_3}{w},\quad \frac{l_4}{w}<1,$$

each of these bounds except the first being optimum, with $1/\sqrt{2}$ conjectured as optimum for it. If no l_2 or no l_3 satisfies the strict inequality, C is a circle.

W. Gustin.

Santaló, L. A. Two characteristic properties of circles on a spherical surface. Math. Notae 11, 73–78 (1951). (Spanish)

Let C be a circular circumference in the plane and K a convex curve inside C which substends at each point of C the constant angle α. The reviewer recently showed [Duke Math. J. 17, 263–267 (1950); these Rev. 12, 123] that if α is an irrational multiple of π or of the form $m\pi/n$, n even, then K is a circle; for other α, curves K other than circles exist. In the present paper the author considers the same problems for curves on the surface of a sphere and shows that in all cases K is a circle. By a polarity consideration he also obtains the dual theorem: a convex curve K on the sphere whose chords of a fixed length λ envelop a circle is a circle. This theorem, unlike its dual, remains valid in the plane.

J. W. Green (Los Angeles, Calif.).

Bourgin, D. G. Restricted separation of polyhedra. Portugaliae Math. 11, 133–136 (1952).

The author proves two theorems on the separation of certain polyhedra in n-space. Theorem 1 (rephrased for the sake of brevity). Let σ and σ' be two disjoint n-simplexes in n-space and suppose the $(n-1)$-faces of σ and σ' can be paired in such a way that the outward normals to corresponding faces have opposite senses. Then σ and σ' can be separated by a hyperplane parallel to a pair of faces. The second theorem concerns an open $(n-1)$-simplex E in n-space consisting of all vectors of the form $(t_1, t_2, \cdots, t_n)$ where $0<t_i<1$ and $\sum_{i=1}^n t_i=1$. Let $N_i=\{t_i|0<\alpha_i<t_i<\beta_i<1\}$. Then $G=\prod_{i=1}^n N_i\cap E$ is an open subset of E. The intervals N_i are called "minimal" for G if $G=\prod_{i=1}^n N_i'\cap E$ implies $N_i\subset N_i'$. Theorem 2. If G and G' are non-trivial open sets in E of the form $\prod N_i\cap E$, $\prod N_i'\cap E$, then, if the neighborhoods for G and G' are minimal, $G\cap G'=\phi$ is equivalent to $N_i\cap N_i'=\phi$ for at least one i.

D. Gale.

Verblunsky, S. On the circumradius of a bounded set. J. London Math. Soc. 27, 505–507 (1952).

Consider a set E in euclidean n-space having the unit sphere as minimal circumsphere: the diameter d of E then satisfies the inequality $d^2\geqq 2(1+n^{-1})$. There are several more or less lengthy known proofs of this result, of which most proceed by first making the easy reduction to the case where E consists of $n+1$ or fewer points lying on the surface of the unit sphere and convexly covering the origin. Unfortunately the author somewhat obscures the simple idea behind his proof of this reduced case by circuitous exposition. A shorter and more direct version follows.

Suppose the origin 0 convexly covered by $n+1$ unit vectors r_α not necessarily distinct. Thus there exist $n+1$ nonnegative numbers λ_α such that (0) $\sum\lambda_\alpha r_\alpha=0$ and (1) $\sum\lambda_\alpha=1$, where $\sum$ denotes summation over all α. Let $d_{\alpha\beta}$ be the distance between r_α and r_β, whence

$$d^2_{\alpha\beta}=(r_\alpha-r_\beta)^2=2-2r_\alpha\cdot r_\beta; \tag{2}$$

and let d be the greatest of the distances $d_{\alpha\beta}$. Since $d\geqq d_{\alpha\beta}$ and $d_{\beta\beta}=0$, it follows from (0), (1), (2) that

$$(1-\lambda_\beta)d^2=\sum{}'\lambda_\alpha d^2\geqq\sum{}'\lambda_\alpha d^2_{\alpha\beta}=\sum\lambda_\alpha d^2_{\alpha\beta}=2, \tag{3}$$

where $\sum'$ denotes summation over all α except β. Summing the end members of (3) over all β and applying (1) yields the sought-for inequality $nd^2\geqq 2(n+1)$. Equality occurs if and only if $d_{\alpha\beta}=d$ for all distinct α, β, whereupon the r_α are vertices of a regular n-simplex.

W. Gustin.

Hodges, J. L., Jr. An extremal problem of geometry. J. London Math. Soc. 26, 311–312 (1951).

Let Γ be a plane convex set, θ a direction in the plane, and Δ a triangle of maximal area inscribed in Γ with one side in the direction θ. It is shown that $\inf|\Delta|/|\Gamma|=3/8$, and that the bound is attained when Γ is a regular hexagon.

M. M. Day (Urbana, Ill.).

Hadwiger, H. Über eine Ungleichung für drei Minkowskische Masszahlen bei konvexen Rotationskörpern. Monatsh. Math. 56, 220–228 (1952).

Let W_ν ($\nu=0, 1, \cdots, k$) be the Minkowskian measures of a convex k-dimensional body V defined by the formula $V(r)=\sum_{\nu=0}^k\binom{k}{\nu}W_\nu r^\nu$ which gives the volume of the parallel body of distance r. A simple proof of the inequality $W_\alpha^{\beta-\gamma}W_\beta^{\gamma-\alpha}W_\gamma^{\alpha-\beta}\geqq 1$ $(0\leqq\alpha<\beta<\gamma\leqq k)$ is given with discussion of the case of equality for convex bodies of revolution. The proof rests on a simple integral representation of W_ν.

L. Fejes Tóth (Veszprém).

Wise, M. E. Dense random packing of unequal spheres. Philips Research Rep. 7, 321–343 (1 plate) (1952).

Let $D(\sigma)$ be the upper bound of the densities of all infinite systems of non-overlapping spheres, of which the logarithms

of the radii have a normal distribution of standard deviation σ. The case $\sigma=0$ corresponds to equal spheres and we have probably $D(0)=18^{-\frac{1}{2}}\pi=0.740\cdots$. A method has been worked out to estimate $D(\sigma)$ from above for different values of σ. The author finds the estimations $D(0)<0.779\cdots$ and $D(0.4)=0.80\cdots$. The first constant equals the density of four equal spheres mutually touching one another in the tetrahedron determined by their centers. The paper makes no claim for rigour. *L. Fejes Tóth* (Veszprém).

Eggleston, H. G. A proof of Blaschke's theorem on the Reuleaux triangle. Quart. J. Math., Oxford Ser. (2) 3, 296–297 (1952).

This note contains a very elementary proof that the Reuleaux triangle has least area among curves of given constant width. *M. M. Day* (Urbana, Ill.).

Santaló, Luis A. Integral geometry in spaces of constant curvature. Repub. Argentina. Publ. Comision Nac. Energia Atomica. Ser. Mat. 1, no. 1, 68 pp. (1952). (Spanish. English summary)

This paper deals with the integral geometry in a space of constant curvature of dimension n. It is divided into two parts, Part I concerned with the densities of linear subspaces and integral formulas derived therefrom and Part II with the kinematic density. These densities are determined explicitly by E. Cartan's method of moving frames. The total measures of several homogeneous spaces are computed, such as the measure of all linear subspaces of a given dimension through a fixed point. Many integral formulas are given, of which we mention the following as samples. 1) The integral over the density of r-dimensional linear subspaces of the area of intersection of such a linear subspace with a fixed regular q-dimensional submanifold C_q is equal to a constant multiple of the area of C_q. This generalizes a well-known formula of Cauchy and it is of interest to notice that it is independent of the curvature of the space. 2) Generalizations of the classical Crofton formulas. 3) Integrals of the elementary symmetric functions of the principal curvatures of a hypersurface and integral formulas for the mean values of such invariants when a hypersurface is cut by a linear subspace. Cases 2) and 3) are carried out only in Euclidean space.

The main result in Part II is the kinematic formula in a space of constant curvature, which contains many formulas of integral geometry as special or limiting cases. *S. Chern* (Chicago, Ill.).

Algebraic Geometry

Chisini, O. Il principio di corrispondenza. Period. Mat. (4) 30, 194–208 (1952).

Godeaux, Lucien. Sur la génération des cubiques planes. Mathesis 61, 258–262 (1952).

Room, T. G. A groupoid of involutory matrices with eight generators. Univ. Washington Publ. Math. 3, 89–98 (1952).

This paper is actually prior to the author's paper in Proc. Cambridge Philos. Soc. 48, 383–391 (1952) [these Rev. 13, 978] which is a sequel to it and gives essentially the same results in a more symmetrical and significant form. On the other hand, the present paper makes clearer the methods by which the results are obtained. The transformations considered here are those of a cubic surface into itself which leave invariant each curve of a pencil of plane sections (with base line v meeting the surface in V_1, V_2, V_3) and interchange the two residual intersections of the surface with each common transversal line of v and a curve p in the surface, of order $3\sigma+1$ and having σ-ple points at V_1, V_2, V_3, so that it meets each cubic of the pencil in one further point. It is shewn that these transformations form a groupoid, the product of any odd number of them being one of them; that the groupoid is generated by eight transformations for which the curves p are respectively the lines $a_1, \cdots, a_6, b_1$ of a double six, and the transform of a_1 by the first of these transformations (this is the residual section of the surface by a ruled cubic with v as double and a_1 as simple directrix and touching the surface at all points of a_1). The surface being represented on a plane by cubics through six points, it is clear that these with V_1, V_2, V_3 are nine associated points and that the transformations here considered are simply those of the paper referred to which are symmetrical with respect to V_1, V_2, V_3. The representation by matrices, similar to that in the paper referred to, is studied, and the results are so similar that they need not be stated here. It may be added that both papers are marred by a large number of misprints which make it in many cases far from clear what the results stated actually are.

P. Du Val (Bristol).

Barsotti, I. Intersection theory for cycles of an algebraic variety. Pacific J. Math. 2, 473–521 (1952).

The object of this paper is to develop an intersection theory of cycles on an algebraic variety, making use of the theory of algebraic correspondences given by the author [Ann. of Math. (2) 52, 427–464, 587 (1950); Trans. Amer. Math. Soc. 71, 349–378 (1951); these Rev. 12, 200; 13, 490]. The geometric notions behind the intersection theory developed here are essentially those used by van der Waerden [Math. Ann. 115, 619–642 (1938)]. Van der Waerden's theory is strictly limited in scope: the ground field is required to be algebraically closed and of characteristic zero, and the quantitative theory of intersections on a variety is confined to varieties without singularities. Barsotti's previous detailed study of correspondences enables him to give a theory valid for arbitrary ground fields, and for cycles on any variety which are locally intersections of the variety with cycles of the ambient space; and the theory is also capable of application to cycles whose point-set intersection is of dimension greater than normal. A detailed statement of the paper is not possible without an elaborate description of the notation used, but the theory includes all the classic results such as Bézout's theorem, and the commutative, associative, and distributive properties of intersections.

W. V. D. Hodge (Cambridge, England).

Vaona, Guido. Classificazione proiettiva delle varietà quasi-asintotiche. Boll. Un. Mat. Ital. (3) 7, 292–298 (1952).

The author shews that, for $k>1$, a quasi-asymptotic V_k or a V_m, of type σ_r, and species t, can be associated with a linear system of algebraic hypersurfaces in S_{k-1}, of order s and dimension $t-1$; and that the projective invariants of such a system lead to a projective classification of the corresponding V_k.

J. A. Todd (Cambridge, England).

Chisini, Oscar. Sulla costruzione a priori delle trecce caratteristiche. Ann. Mat. Pura Appl. (4) **33**, 353–366 (1952).

Let $F_\lambda(x, y, z)=0$ be an algebraic surface, varying in a family depending on a parameter λ, and let $\phi_\lambda(x, y)=0$ be the branch curve in the (x, y)-plane of the function $z(x, y)$ defined by F_λ. If F_0 has a double curve γ^* whose projection $\gamma(x, y)=0$ counts doubly as a part of ϕ_0 (i.e., $\phi_0=\gamma^2\psi$), γ^2 and ψ will be connected at points of contact of γ and ψ each of which is the limit of three cusps of ϕ_λ. With this in mind, this paper considers the problem of a variable plane curve (not necessarily a branch curve), with three cusps, which, in the limit, acquires a double component touching the residual part at a point O; O being the common limit of the three cusps. The author finds the contribution, associated with the point O, to the braids (trecce caratteristiche) associated with the limiting curve. [Cf. Chisini, Ist. Lombardo Sci. Lett. Rend. (2) **66**, 1141–1155 (1933); and M. Dedò, ibid. (3) **14(83)**, 227–258 (1950); these Rev. **13**, 973.] *D. B. Scott* (London).

Blaschke, Wilhelm. Connessioni fra varietà di C. Segre e la geometria dei tessuti. Boll. Un. Mat. Ital. (3) **7**, 259–261 (1952).

The Segre variety V_3 in S_7 representing the product of three lines can be regarded, in three ways, as the product of a line and a quadric, and thus contains three ∞^1 systems of quadric surfaces. A general prime section is thus a surface containing three ∞^1 systems of conics; it is remarked that these form a hexagonal web (Sechseckgewebe) as defined, for instance, by Blaschke and Bol [Geometrie der Gewebe, Springer, Berlin, 1938]. *D. B. Scott* (London).

Differential Geometry

Zahorski, Zygmunt. Sur les courbes dont la tangente prend sur tout arc partiel toutes les directions. Czechoslovak Math. J. **1(76)** (1951), 105–117 (1952) = Čehoslovack. Mat. Ž. **1(76)** (1951), 125–139 (1952).

Théorème I: Si une courbe $\mathfrak{C}$ de l'espace euclidien possède une tangente (orientée ou non orientée) sauf aux points d'un ensemble dénombrable, cette tangente ne peut prendre deux directions θ' et θ'' données d'avance sur tout arc partiel. Théorème II: Il existe un arc simple et rectifiable $\mathfrak{S}$ dont la tangente orientée prend sur chaque arc partiel toutes les directions. Idée des démonstrations: I. $P=f(t)$, $a\leqq t\leqq b$, étant une représentation paramétrique de $\mathfrak{C}$ sans intervalle de constance, l'ensemble $T(\theta)$ des valeurs de t où la tangente a une direction déterminée θ est un G_δ. Si pour $\theta'\neq\theta''$, $T(\theta')$ et $T(\theta'')$ sont denses sur $[a, b]$, ils ont une valeur en commun où la tangente ne peut exister. II. $\alpha(t)$: fonction singulière de Lebesgue; $e(s)$: application continue de l'intervalle $[0, 1]$ sur la sphère unité telle que $e(0)=e(1)$; $P=\int_0^t e(\alpha(s))ds$. La courbe $\mathfrak{C}$ décrite par P est un arc simple possédant partout une tangente orientée qui, sur l'ensemble de Cantor de $[0, 1]$, prend toutes les directions. $\mathfrak{S}$ est construite à partir de $\mathfrak{C}$ par une méthode de condensation de singularités. [Remarque du rapporteur: Théorème I est un corollaire d'un théorème de G. Choquet [J. Math. Pures Appl. (9) **26**, 115–226 (1948); ces Rev. **9**, 419] ou de R. Brisac [C. R. Acad. Sci. Paris **224**, 257–258 (1947); ces Rev. **8**, 321] d'après lesquels sur un résiduel de $[a, b]$ le contingent de $\mathfrak{C}$ est identique au paratingent. En effet si sur tout arc partiel de $\mathfrak{C}$ la tangente prend les directions θ' et θ'', le paratingent contiendra partout ces directions ce qui sera impossible aux points de coincidence avec un contingent ne comprenant qu'une direction.] *C. Pauc* (Nantes).

Löbell, Frank. Variation von Kurvenintegralen über Linienelementfunktionen. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1951**, 1–9 (1952).

If ϕ is a line element function on a surface, consider the integral $\int_c\phi ds$ where c is a curve on the surface and ϕ is evaluated at each point for the direction of c. A formula is derived for the variation of this integral induced by a variation of the curve c. This formula is in terms of geodesic derivatives of ϕ. The formula is then applied to a number of special cases of geometric interest such as the case where ϕ represents the normal curvature of c or the geodesic curvature vector of c. The last three applications arise from the theory of surface mappings. *S. B. Jackson.*

Löbell, Frank. Integrabilitätsbedingungen in der Theorie der Flächenabbildungen. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1951**, 11–28 (1952).

Consider a mapping $y\leftrightarrow x$ of two surfaces in Euclidean 3-space. Associated with such a mapping are three quantities previously introduced by the author; n the projection measure (Rissmassstab), q the transverse measure (Querrissmassstab), and γ the normal measure (Normalrissmassstab) [same S.-B. **1947**, 15–23 (1949); these Rev. **11**, 130]. The integrability conditions that must be satisfied by n, q, and γ as well as the normal curvature and geodesic torsion of x are obtained. All five of these quantities are line element functions and the concept of geodesic derivative enables the relations to be written in simpler form. Invariant and vector forms of these relations are also considered in some detail. The familiar Codazzi-Mainardi equations are a special case of the relations obtained, and are obtained by considering the mapping of x on the unit sphere so that corresponding points have parallel normals. Interestingly enough, the Gauss equation cannot be so obtained however. *S. B. Jackson* (College Park, Md.).

Jonas, Hans. Die Scherksche Minimalfläche als Gegenstand einer anschaulichen geometrischen Deutung des Additionstheorems für das elliptische Integral 1. Gattung. Math. Nachr. **8**, 41–52 (1952).

In a suitable coordinate system, H. F. Scherk's minimal surfaces can be given the parametric equations

$$(1)\qquad \begin{aligned} x&=\ln(1+\sin\varphi\sin\psi)-\ln(1-\sin\varphi\sin\psi),\\ y&=\sin\gamma\cdot 2\tan^{-1}(\cos\varphi\cdot\tan\psi),\\ z&=\cos\gamma\cdot 2\tan^{-1}(\tan\varphi\cos\psi).\end{aligned}$$

Here, the parameter γ distinguishes the various surfaces and the parameter curves $\varphi=$const. and $\psi=$const. are asymptotic lines. Lie proved that (1) is the locus of the midpoints of the chords of any one of its skew asymptotic lines [Gesammelte Abhandlungen, Bd. I, Teubner, Leipzig, 1934, pp. 414–439]. The author shows: Suppose two points move at constant speed on such a line. If they move in the same direction [in opposite directions], then their midpoint moves on an asymptotic line of the same [the other] family. "Von besonderer Bedeutung ist der Umstand, dass zugleich der bekanntlich sehr enge Kreis derjenigen Anwendungen des Eulerschen Additionstheorems erweitert wird, bei denen das elliptische Integral erster Gattung in der anschaulichen Bedeutung einer Bogenlänge auftritt."

By a parallel projection, (1) is mapped on

$$(2)\qquad x=\ln\frac{1+\sin\varphi\sin\psi}{1-\sin\varphi\sin\psi},\quad y=2\tan^{-1}(\cos\varphi\tan\psi)\ [z=0].$$

Since (2) can be interpreted as the limit case $\gamma=\pi/2$ of (1), the author's result remains valid for the curves $\varphi=$const. and $\psi=$const. of (2). They form an orthogonal net. Through

$$x\leftrightarrow x'=\sin\varphi/\sin\psi,\quad y\leftrightarrow y'=\cos\varphi\cot\psi$$

it is mapped on a net of confocal conics. The two nets have parallel tangents in corresponding points. *P. Scherk.*

Terracini, Alessandro. Osservazioni sulle linee principali di alcune classi di superficie dello spazio a cinque dimensioni. Boll. Un. Mat. Ital. (3) 7, 247–252 (1952).

L'auteur étudie une relation entre l'ordre de multiplicité d'un système de lignes principales d'une surface S de S_5, et l'ordre d'approximation σ de l'incidence des plans tangents en deux points infiniment voisins d'une ligne principale du système. Il montre que si, pour S, un système de lignes principales est au moins triple, l'ordre d'approximation en question, σ, est $\geqq 6$. D'autre part si, sur une surface S de S_5, les plans tangents le long des lignes principales d'un système (supposées non planes) sont telles que l'incidence de deux plans infiniment voisins soit d'ordre $\sigma\geqq 6$, et si le système des lignes principales envisagé est au moins double, il est nécessairement triple. L'auteur montre sur un exemple (surface S du type W), que dans l'hypothèse d'un système au moins triple, la condition $\sigma\geqq 6$ ne peut être améliorée (on n'a pas nécessairement $\sigma\geqq 8$). Et l'exemple envisagé conduit à la conséquence que le fait, pour un système de lignes principales, d'être quadruple (et non seulement triple), n'entraîne pas que l'ordre d'approximation de l'incidence des plans tangents infiniment voisins à la surface le long d'une ligne principale soit $\geqq 8$. *P. Vincensini.*

Džavadov, M. A. Conformal transformations in Euclidean and pseudo-Euclidean spaces of an arbitrary number of dimensions as linear fractional transformations. Doklady Akad. Nauk SSSR (N.S.) 86, 653–656 (1952). (Russian)

It is a classical result due to E. Study that the conformal transformations of a Euclidean space R_3 or R_4 can be represented by linear fractional transformations of quaternions; in the case of R_3 these transformations leave a hypersurface of the space invariant. The present paper deals with a generalization of this idea. It first considers a pseudo-Euclidean space ${}^{n-1}R_{2n-1}$ whose metric is

$$(x,x)=\sum_{i=1}^{2n-1}(-1)^{i-1}(x^i)^2.$$

With each point of this space is associated a matrix $X^{(n)}$ of order 2^{n-1} defined recursively by

$$X^{(n)}=\begin{pmatrix} X^{(n-1)} & (x^{2n-2}+x^{2n-1})E^{(n-1)} \\ (-x^{2n-2}+x^{2n-1})E^{(n-1)} & -X^{(n-1)} \end{pmatrix}$$

where $E^{(n-1)}$ is a unit matrix of order 2^{n-1}. Thus

$$X^{(2)}=\begin{pmatrix} x^1 & x^2+x^3 \\ -x^2+x^3 & -x^1 \end{pmatrix}.$$

The stated theorem is that the group of conformal transformations of ${}^{n-1}R_{2n-1}$ is isomorphic to the group of linear fractional transformations $w=(As+B)(Cs+D)^{-1}$ in the algebra of real matrices of order 2^{n-1} leaving invariant the subspace of matrices of the type $X^{(n)}$. For a Euclidean R_{2n-1} or for a pseudo-Euclidean ${}^lR_{2n-1}$ the same theorem holds except that x^{2k} is replaced by ix^{2k} in the Euclidean case or, in the case of ${}^lR_{2n-1}$, x^{2k} that occurs with a $+$ sign in the metric is replaced by ix^{2k}. In the case of spaces of even number of dimensions for ${}^nR_{2n}$ with metric $(x,x)=\sum_{k=1}^{2n}(-1)^{k-1}(x^k)^2$, the associated matrix is of the form $X^{(n)}(e)=X^{(n)}+ex^{2n}E^{(n)}$ where $X^{(n)}$ and $E^{(n)}$ are defined as above and 1, e are elements of a Clifford algebra. As above in case of Euclidean space x^{2k} is replaced by ix^{2k}. The group of conformal transformations of such a space is isomorphic to the group of linear fractional transformations $w=(Az+B)(Cz+D)^{-1}$ and $w=(A\bar{z}+B)(C\bar{z}+D)^{-1}$. The author observes that the above groups for R_{2n-1}, R_{2n}, ${}^lR_{2n-1}$, and ${}^lR_{2n}$ are spinor representations of motions in non-Euclidean spaces ${}^1S_{2n}$, ${}^1S_{2n+1}$, ${}^{l+1}S_{2n}$, and ${}^{l+1}S_{2n+1}$.

M. S. Knebelman (Pullman, Wash.).

Rozenfel'd, B. A. The projective differential geometry of the family of pairs P_m+P_{n-m-1} in P_n. Amer. Math. Soc. Translation no. 77, 32 pp. (1952).

Translated from Mat. Sbornik N.S. 24(66), 405–428 (1949); these Rev. 11, 133.

Guggenheimer, H. Über vierdimensionale Einsteinräume. Experientia 8, 420–421 (1952).

On a four-dimensional compact complex analytic manifold there is a two-dimensional characteristic homology class with integral coefficients, first introduced by the reviewer. Let ψ be its self-intersection and χ the Euler-Poincaré characteristic of the manifold. The author proves that, if the manifold has an Einstein metric, then the inequality $0\leqq\psi\leqq 3\chi$ holds. If the metric is of positive curvature, the manifold must have the same additive homology structure as the complex projective plane. *S. Chern.*

Takano, Kazuo. Geodesic torsions and geodesic curvatures in Riemannian spaces. Rep. Univ. Electro-Commun. 1951, no. 3, 71–74 (1951). (Japanese summary)

Associated to the $(\alpha-1)$-normal $(\alpha=2,\cdots,n)$ at a point of a curve on a hypersurface in an n-dimensional Riemannian space, there are defined two scalars T_α and $\overset{*}{C}_\alpha$ which can respectively be considered as generalizations of the geodesic torsion and the geodesic curvature of a curve on a surface in a three-dimensional Euclidean space. These T_α's were expressed in a different form by A. Kawaguchi and T. Hosokawa [Tôhoku Math. J. 37, 340–346 (1933)].

C.-C. Hsiung (Bethlehem, Pa.).

Hsiung, Chuan-Chih. Some curves in Riemannian space. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 38, 816–823 (1952).

Soit $\bar{C}$ une courbe voisine d'une courbe donnée C dans un espace riemannien à n dimensions, telle que le point $\bar{P}$ de $\bar{C}$ correspondant à un point quelconque P de C se trouve sur le vecteur unitaire $\xi_{(2)}$ de la seconde normale à C en P et à une distance infinitésimale ϵ de P. Soit $\bar{\xi}'_{(2)}$ le vecteur obtenu à partir du vecteur unitaire $\bar{\xi}_{(2)}$ de la seconde normale à $\bar{C}$ en $\bar{P}$ en faisant subir à ce dernier un déplacement parallèle infinitésimal le long de $\xi_{(2)}$ jusqu'en P. On obtient $n-2$ conditions nécessaires et suffisantes pour que $\bar{\xi}'_{(2)}$ coincide avec $\xi_{(2)}$ au premier ordre près en ϵ. (Author's summary.) *V. Hlavatý* (Bloomington, Ind.).

Kosambi, D. D. Path geometry and continuous groups. Quart. J. Math., Oxford Ser. (2) 3, 307–320 (1952).

The author describes the content of this paper as "an attempt to show the relationship between topological groups and the differential geometry of path-spaces." The treatment is given in several stages. A. The author claims to prove that a restricted path-space (given by equations $\ddot{x}+\alpha(x,\dot{x})=0$) can be considered locally as a neighbourhood

of the identity of a topological group, the paths corresponding to the one-parameter subgroups. B. The author discusses the possibility of introducing a metric tensor in a topological group such that the one-parameter subgroups correspond to the geodesics of this metric. C. A summary of some well-known results about Lie groups is followed by a number of theorems about projectively flat path-spaces, in which the Weyl tensor plays an important rôle. A corollary states that the paths of any symmetric linear connection are also the geodesics of a Riemannian space. Some of the arguments do not seem to be very conclusive. *A. Nijenhuis.*

Tashiro, Yoshihiro. Note sur la dérivée de Lie d'un être géométrique. Math. J. Okayama Univ. 1, 125–128 (1952).

In einem n-dimensionalen Raum sei ein geometrisches Objekt Ω^A gegeben und $X\Omega^A$ bezeichne seine in Bezug auf die infinitesimale Transformation $\bar{x}^\lambda = x^\lambda + \xi^\lambda \delta t$ gebildete Liesche Ableitung. Ist G_r eine Liesche Transformationsgruppe und sind $\xi_a{}^\lambda$ $(a=1, \cdots, r)$ ihre Geschwindigkeitsfelder, so findet Verf. für die zu ihnen gehörigen Lieschen Ableitungen X_a die Beziehung $(X_bX_c)\Omega^A = c^a{}_{bc}X_a\Omega^A$ in derselben bedeutet (X_bX_c) den zu den Lieschen Ableitungen gehörigen Kommutator. *O. Varga* (Debrecen).

Tonowoka, Keinosuke. A generalization of Cartan space. J. Math. Soc. Japan 4, 134–145 (1952).

The generalization considered is one in which the fundamental integrand of the k-fold integral

$$\int_k F(u^a, x^i, \partial x^i/\partial u^a, \partial^2 x^i/\partial u^a \partial u^b) du^1 \cdots du^k$$

involves the second derivatives of the x coordinates with respect to the parameters u of the k-dimensional surface. After giving some preliminary identities, the author proceeds to obtain some quantities which are intrinsic in that they satisfy the tensor law of transformation with respect to both coordinate and parameter transformations. In the first paragraph the fundamental tensors of the space are introduced as well as generalizations of the Synge vectors. Later the notions of covariant differential and covariant derivative are generalized. *E. T. Davies.*

Varga, O. Eine geometrische Charakterisierung der Finslerschen Räume skalarer und konstanter Krümmung. Acta Math. Acad. Sci. Hungar. 2, 143–156 (1951). (Russian summary)

After explanation in §1 of the curvatures in a Finsler space, the author proves the following theorems. (1) For a Finsler space of scalar curvature [Berwald, Math. Z. 25, 40–73 (1926)] it is necessary and sufficient that by parallel displacement of the normalized line element l^α around an infinitesimal circuit the difference vector between the original l^α and its displaced one is a linear combination of l^α and of the two vectors $t_1{}^\alpha$ and $t_2{}^\alpha$ of the 2-direction of the circuit. (2) When ψ_α is an arbitrary covariant vector such that $\psi_\alpha l^\alpha = 0$ and $R(x, v)$ any scalar, the Finsler space whose affinor $R_{\beta\delta}{}^\alpha$ has the form

$$\delta_\beta{}^\alpha(\psi_\delta + Rl_\delta) - \delta_\delta{}^\alpha(\psi_\beta + Rl_\beta) + (l_\delta\psi_\beta - l_\beta\psi_\delta)l^\alpha$$

is of scalar curvature and the curvature-measure (Krümmungsmass) is $R(x, v)$, where $l^\alpha = v^\alpha/L(x, v)$. (3) For a Finsler space of constant curvature it is necessary and sufficient that the difference vector stated in (1) belongs to the 2-direction of the circuit. (4) When b_α is an arbitrary covariant vector, the Finsler space whose affinor $R_{\beta\delta}{}^\alpha$ is given by $\delta_\beta{}^\alpha b_\delta - \delta_\delta{}^\alpha b_\beta$ is of constant curvature. *A. Kawaguchi* (Sapporo).

***Rund, Hanno. A theory of curvature in Finsler spaces.** Colloque de Topologie de Strasbourg, 1951, no. IV, 12 pp. La Bibliothèque Nationale et Universitaire de Strasbourg, 1952.

This is a summary of a theory of curvatures in Finsler spaces. Most of the results have since been published in detail in other papers [1) Thesis, Cape Town, 1950; 2) Arch. Math. 3, 60–69 (1952); 3) Math. Z. 54, 115–128 (1951); 4) Math. Ann. 125, 1–18 (1952); these Rev. 13, 987, 159; 5) the paper reviewed second below]. The main idea is a notion of parallelism different from the classical ones of Berwald and Cartan. It has the advantage of depending only on the point and not on the direction, but the disadvantage that the length of a vector is not preserved. In a sense the resulting curvature measures the difference of a Finsler metric from the tangent Minkowskian metric. The author shows how various notions in the differential geometry of Finsler spaces can be established on the basis of this parallelism. *S. Chern* (Chicago, Ill.).

Rund, Hanno. Eine Krümmungstheorie der Finslerschen Räume. Math. Ann. 125, 1–18 (1952).

This is one of a series of papers of the author in his development of a theory of curvatures of Finsler spaces [see the preceding review]. The parallel displacement depends only on the point and not on the direction. Curvature is defined by computing the geodesic deviation, and the curvature tensor in the sense of the author enters naturally in the Jacobi differential equations of the variational problem arising from the given Finsler metric. The two-dimensional case is considered in more detail. Among other results the author introduces a curvature scalar which generalizes the Gaussian curvature in Riemannian geometry and interprets it by means of formulas analogous to those of Bertrand and Puiseux. A generalization of the Gauss-Bonnet formula in two dimensions is established. *S. Chern.*

Rund, Hanno. The theory of subspaces of a Finsler space. I. Math. Z. 56, 363–375 (1952).

This paper develops the analytical foundations of the theory of subspaces of a Finsler space by taking as starting-point a Hamiltonian function defined in the subspace and applying to it the covariant differentiation introduced by the author in an earlier paper [Math. Z. 54, 115–128 (1951); these Rev. 13, 159]. Formulas are obtained for the comparison of the induced and intrinsic covariant derivations. In the case of hypersurfaces a normal cone is introduced to take the place of normal vector, and several theorems of classical differential geometry are thus generalized. *S. Chern.*

Nasu, Yasuo. On the torse-forming directions in Finsler spaces. Tôhoku Math. J. (2) 4, 99–102 (1952).

A torse-forming vector field is one such that when the space is developed along a curve, the lines along the vectors of the field form a torse (i.e., a developable surface). This concept applies to a projectively connected space. It is proved that if a Finsler space admits a torse-forming vector field, there exists a one-parameter family of totally umbilical hypersurfaces whose trajectories are geodesics. Normal form of the metric of such a Finsler space is given. *S. Chern* (Chicago, Ill.).

Owens, O. G. The integral geometry definition of arc length for two-dimensional Finsler spaces. Trans. Amer. Math. Soc. 73, 198–210 (1952).

Ist C^* eine geschlossene konvexe Kurve der Euklidischen Ebene, dann ist ihr Umfang gleich dem Geradenmass seiner Treffgeraden. Fällt man vom Ursprung eines Descartschen Koordinatensystems das Lot auf eine Gerade, bezeichnet Θ den Winkel desselben mit der positiven x-Achse und p die Länge desselben, so kann dieses Resultat in der Form $\oint_{C^*} ds = \int\int_D dp d\Theta$ ausgedrückt werden. Das Doppelintegral ist dabei über alle Treffgeraden von C^* zu erstrecken. Für den Fall einer zweidimensionalen Finslerschen Mannigfaltigkeit mit dem Bogenelement $ds = F(x, y, dx, dy)$ findet Verf. eine entsprechende allgemeinere Formel

$$\oint_{C^*} F(x, y, dx, dy) = \int\int_D \sigma(\Theta, p) dp d\Theta.$$

Jetzt bedeutet C^* eine geschlossene Kurve die extremalkonvex ist, und der Integrationsbereich des Doppelintegrales ist analog wie oben zu erklären. Die Existenz einer extremalkonvexen geschlossenen Kurve wird durch geeignete Feldbedingungen gesichert. Eine Extremale der zugrunde liegenden Schar wird als Transversale derjenigen Extremalen bestimmt, die mit der positiven x-Achse den Winkel Θ einschliesst, und p bedeutet die Euklidische Bogenlänge letzterer Extremalen bis zum Schnittpunkt mit der Transversalen. $F(x, y, dx, dy)$ soll für sämtliche Argumente dreimal stetig differenzierbar sein. Es sei bemerkt dass W. Blaschke nachgewiesen hat [Abh. Math. Sem. Univ. Hamburg 11, 359–366 (1936)], dass die Länge von C^* (in der Finslerschen Massbestimmung) gleich dem Mass der C^* schneidenden Extremalen ist. *O. Varga* (Debrecen).

***Hodge, W. V. D. The theory and applications of harmonic integrals.** 2d ed. Cambridge, at the University Press, 1952. x+282 pp. 27 s. 6 d., $5.50.

For the most part this is a reprint by photo-offset of the first edition [Cambridge, 1941; these Rev. 2, 296]. The proof of the existence theorem for harmonic integrals has been corrected, §26.3 on the relation between products of forms and intersections of cycles has been replaced, and a list of important papers dealing with harmonic integrals which appeared since the first edition is appended.

Fumi, F. G. Matter tensors in symmetrical systems. Nuovo Cimento (9) 9, 739–756 (1952).

The known representations of the finite rotation groups are used to find sets of independent components for Cartesian tensors of specified symmetry character. Tables of such components are given for third and fourth order tensors and pseudotensors. *A. J. Coleman* (Toronto, Ont.).

NUMERICAL AND GRAPHICAL METHODS

***Jones, C. W. A short table for the Bessel functions**

$$I_{n+1/2}(x), \quad \frac{2}{\pi} K_{n+1/2}(x).$$

Prepared on behalf of the Mathematical Tables Committee of the Royal Society. Cambridge, at the University Press, 1952. 20 pp. $1.25.

In the preface to the recent, final, volume of the British Association Mathematical Tables series, Professor Bickley mentioned that the Mathematical Tables Committee of the Royal Society is proceeding with the tabulation of Bessel functions of non-integer orders. The present publication is a first installment of such tables and it is hoped that it will be followed by more extensive tables.

The principal tables of the present pamphlet give

$$x^{-n-1/2} I_{n+1/2}(x) \quad \text{and} \quad 2\pi^{-1} x^{n+1/2} K_{n+1/2}(x)$$

for $x = 0(.1)5$, $n = 0(1)10$, and $e^{-x} I_{n+1/2}(x)$ and $2\pi^{-1} e^x K_{n+1/2}(x)$ for $x = 5(.1)10$, $n = 0(1)10$. Most values are given to seven or eight significant figures, with modified second central differences, and it is believed that the tabular values are correct to within 0.5 unit of the final decimal place. Supplementary tables give $x^{1/2}$ and $(\frac{1}{2}\pi x)^{1/2}$ for $x = 0(.1)5$, e^{-x} and $\frac{1}{2}\pi e^{-x}$ for $x = 5(.1)10$, e^{-x} for $x = 0(.01).1$, $2\pi^{-1} x^{n+1/2} K_{n+1/2}(x)$ for $x = 5(.1)6$, $n = 7(1).10$, and coefficients of the polynomials which occur in the expression of $I_{n+1/2}$ and $K_{n+1/2}$ in terms of elementary functions. *A. Erdélyi.*

Clemmow, P. C., and Munford, Cara M. A table of

$$\sqrt{(\tfrac{1}{2}\pi)}\, e^{\frac{1}{2}i\pi\rho^2} \int_\rho^\infty e^{-\frac{1}{2}i\pi\lambda^2} d\lambda$$

for complex values of ρ. Philos. Trans. Roy. Soc. London. Ser. A. 245, 189–211 (1952).

The author tabulates the function of the title for $\arg \rho = 0°(1°)45°$, $|\rho| = 0.00(.01)0.80$, 4D. There is a short discussion of the reasons for the selection of this function rather than related ones, of the method of computation, of several physical problems in which this function occurs, and of related tables. *J. V. Wehausen.*

Barfield, W. D., and Broyles, A. A. Coulomb functions for heavy nuclear particles. Physical Rev. (2) 88, 892 (1952).

Coulomb wave functions have been tabulated recently by Bloch et al. [Rev. Modern Physics 23, 147–182 (1951); these Rev. 13, 234], and by the National Bureau of Standards [Tables of Coulomb wave functions, vol. I, Nat. Bur. Standards Appl. Math. Ser., no. 17, Washington, D. C., 1952; these Rev. 13, 988]. Neither of these tabulations include high values of η. For large η, the asymptotic formulas are very useful, except near the transition point. For this reason, the authors provide a short table of $F_0(2\eta)$, $F_0'(2\eta)$, $G_0(2\eta)$ for $\eta = 10(5)40(10)200$. They also comment on the numerical computation of Coulomb functions in this range. *A. Erdélyi* (Pasadena, Calif.).

Lubkin, Samuel. A method of summing infinite series. J. Research Nat. Bur. Standards 48, 228–254 (1952).

This paper is a development of the idea that if the terms of a series $\sum a_n$ are such that the ratios $R_n = a_n/a_{n-1}$ satisfy appropriate conditions, then the number

$$t_n = a_0 + a_1 + \cdots + a_{n-1} + a_n \left[1 + \left(\frac{a_{n+1}}{a_n}\right) + \left(\frac{a_{n+1}}{a_n}\right)^2 + \cdots\right]$$

is a better approximation to the value V of the series than $s_n = a_0 + a_1 + \cdots + a_n$ is. Iteration of the idea is obtained by applying it to the series $\sum b_n$ with partial sums t_n. Relevant motivations, theorems, and calculations are given. In addition to recent works cited by the author, a contemporary paper by E. Pflanz [Arch. Math. 3, 24–30 (1952); these Rev. 14, 321] treats the same subject. A more serious bibliographic omission is a paper of Kummer [J. Reine Angew. Math. 16, 206–214 (1837)] which contains ideas

that are now being partially revived in more than one computing center. It seems that this excellent paper of Kummer, which claims priority by the title "Eine neue Methode, die numerischen Summen langsam convergirender Reihen zu berechnen", has not be adequately publicized by modern textbooks. *R. P. Agnew* (Ithaca, N. Y.).

Kašanin, R. Interprétation géométrique du schéma de Banachiewicz. Srpska Akad. Nauka. Zbornik Radova 18, Matematički Inst. 2, 93–96 (1952). (Serbo-Croatian. French summary)

A nonsingular matrix A can always be written $L^{-1}U$, where L [where U] is a lower [upper] triangular matrix. The author interprets A as transforming the coordinates of a vector in one oblique basis $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3$ to those in another, $\mathfrak{b}_1, \mathfrak{b}_2, \mathfrak{b}_3$. Consider a third basis $\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{c}_3$ such that $\mathfrak{c}_1 \| \mathfrak{b}_1$, $\mathfrak{c}_3 \| \mathfrak{a}_3$, while $\mathfrak{c}_2$ lies in the plane of $\mathfrak{b}_1, \mathfrak{b}_2$ and in the plane of $\mathfrak{a}_2, \mathfrak{a}_3$. Then L [then U] is the matrix expressing the $\mathfrak{c}$'s in terms of the $\mathfrak{a}$'s [$\mathfrak{b}$'s]. *G. E. Forsythe* (Los Angeles, Calif.).

Angelitch, Tatomir. Résolution des systèmes d'équations linéaires algébriques par la méthode de Banachiewicz. Srpska Akad. Nauka. Zbornik Radova 18, Matematički Inst. 2, 71–92 (1952). (Serbo-Croatian. French summary)

This is a very clear and leisurely matrix exposition of unabridged Gaussian elimination and the theoretically equivalent "unsymmetrical Cholesky" = Banachiewicz method for solving linear systems. There are several numerical illustrations. The presentation is largely based on that in section 23 of Zurmühl [Matrizen . . . , Springer, Berlin, 1950; these Rev. 12, 73]. The last equation of the summary should read $b_{ij} = -c_{ji}/c_{ii}$. *G. E. Forsythe* (Los Angeles, Calif.).

Kikukawa, Makoto. A numerical method for multiplication, reciprocation and division of matrices and for the solution of simultaneous linear algebraic equations. Tech. Rep. Osaka Univ. 2, 11–30 (1952).

Largely independently of Western work the author has developed the "unsymmetrical Cholesky" method for inverting a matrix, evaluating a determinant, and solving a system $Af = c$. The exposition seems correct and fairly readable, using matrices and discussing the often neglected situation of a zero divisor. About 20 intricate drawings describe the computing schemes. See section 23 of Zurmühl [Matrizen, cf. the preceding review] for an exposition of the material with references back to Benoit (1924) and Banachiewicz (1937). *G. E. Forsythe.*

Michkovitch, V. V. Résolution des systèmes d'équations linéaires algébriques à l'aide des cracoviens. Srpska Akad. Nauka. Zbornik Radova 18, Matematički Inst. 2, 53–70 (1952). (Serbo-Croatian. French summary)

Cracovians, devised and long advocated by Banachiewicz [for references see Zurmühl, Matrizen, cf. the second preceding review], differ from matrices in that multiplication is "column-by-column." The author expounds their algebra, including the Cholesky-Banachiewicz result that a nonsingular cracovian can be expressed as the product of two upper triangular cracovians. The theorem is applied to the numerical solution of linear systems. *G. E. Forsythe.*

Voetter, Heinz. Über die numerische Berechnung der Eigenwerte von Säkulargleichungen. Z. Angew. Math. Physik 3, 314–316 (1952).

For the numerical calculation of the secular determinant (characteristic polynomial) $|A-xI|$ of any matrix A of order n, the author proposes [in the reviewer's notation] n successive left multiplications of $A-xI$ by matrices of type

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ a & b & c & d & \cdots & ex+f \end{bmatrix},$$

where $a, b, \cdots$ are scalars and x is an indeterminate. The resulting matrix has zeros below the main diagonal and scalars on the diagonal, except that the diagonal element in the last row is $|A-xI|$. The choice of $a, b, \cdots$ is clear from the formulas and numerical example given for $n=4$. The algorithm is said to be convenient and fast for desk calculators, and to require no intermediate recordings. *G. E. Forsythe* (Los Angeles, Calif.).

Krasnosel'skiĭ, M. A., and Kreĭn, S. G. Remark on the distribution of errors in the solution of a system of linear equations by means of an iterative process. Uspehi Matem. Nauk (N.S.) 7, no. 4(50), 157–161 (1952). (Russian)

Let the positive definite symmetric matrix A have eigenvalues $0<\lambda_1<\cdots<\lambda_n<1$. In the ordinary iterative process to find the solution x^* of the system $x = Ax+b$, one picks x_0 arbitrarily and forms $x_{m+1} = Ax_m + b$ $(m=0, 1, \cdots)$. Let G be the sphere with center 0 and radius α. Suppose one terminates the iteration when $\delta_p \equiv x_{p+1} - x_p \in G$, but δ_{p-1} non-$\in G$. Thus p depends on x_0. Then, since the error $\epsilon_p \equiv x^* - x_p$ is $(I-A)^{-1}\delta_p$, one has the usual error estimate that (*) $\|\epsilon_p\| \leq \alpha(1-\lambda_n)^{-1}$. The authors prove a useful theorem showing that (*) is likely to be nearly an equality, namely: Suppose ϵ_0 is uniformily distributed in the sphere of radius R. Then, for each $\eta<1$, the probability

$$\text{Prob } \{\eta\alpha\lambda_n(1-\lambda_n)^{-1} \leq \|\epsilon_p\| \leq \alpha(1-\lambda_n)^{-1}\}$$

tends to 1 as $R\to\infty$. In the proof the authors estimate the volumes of sets associated with the ellipsoids $A^{-m}(I-A)^{-1}G$ $(m=-1, 0, \cdots, p)$. *G. E. Forsythe.*

Lanczos, Cornelius. Solution of systems of linear equations by minimized-iterations. J. Research Nat. Bur. Standards 49, 33–53 (1952).

Das bereits insbesondere in einer früheren Arbeit [Lanczos, dasselbe J. 45, 255–282 (1950); diese Rev. 13, 163] zur Berechnung von Eigenwerten aufgestellte und diskutierte Verfahren der minimalisierenden Iteration wird hier in einer etwas modifizierten Form zur Lösung eines linearen Gleichungssystems

$$(1) \qquad Ay = b_0$$

herangezogen. Seien p_k und q_k definiert durch

$$p_k = [A^k - (a_0 + a_1A + \cdots + a_{k-1}A^{k-1})]b_0,$$
$$q_k = [1 - (\bar{a}_1A + \cdots + \bar{a}_kA^k)]b_0$$

und p_k^* und q_k^* mittels der transponierten Matrix A^* durch die entsprechenden Formeln, wobei die a bzw. die $\bar{a}$ so bestimmt sein sollen, dass $p_kp_k^*$ bzw. $q_kq_k^*$ zum Minimum gemacht werden, dann besteht die $p-q$ Methode (vom Verf. als Methode II bezeichnet) zur Auflösung des Systems (1) darin, dass y nach den $q_k = -1/\bar{a}_kq_k$ entwickelt wird. Für die p_k, q_k und die Entwicklungskoeffizienten ergeben sich Rekursionsformeln. Dieses Verfahren ist für grosse n sehr empfindlich gegenüber Abrundungsfehler. Um diesen Nachteil zu beseitigen, wird noch ein zweites Verfahren (Methode I) besprochen, das im wesentlichen auf den Minimaleigen-

schaften der auf das Intervall (0, 1) reduzierten Tschebyscheff-Polynome, $T_n(x)=\cos n\vartheta$, $x=\frac{1}{2}(1-\cos\vartheta)=\sin^2\frac{1}{2}\vartheta$, beruht. Zur Durchführung dieser Reduktion betrachtet der Verfasser jene quadratische semidefinite Form Q mit der Matrix A_1 von $n+1$ Veränderlichen, die man erhält, wenn man die den einzelnen Zeilen von (1) entsprechenden Linearformen quadriert und summiert und dann durch die grösste Wurzel λ_M der charakteristischen Gleichung dividiert. Die grösste charakteristische Zahl λ_M der Matrix $A_2=\lambda_M I-A_1$ entspricht dann dem Eigenvektor für den Q verschwindet. Praktisch benötigt man nur eine rohe von Geršgorin aufgestellte Schranke für λ_M. Bei diesem Verfahren wird der mit einem passend gewählten Normierungsfaktor multiplizierte Unterschied der rechten und linken Seite von (1) als "Restvektor" r_k bezeichnet. Das Verfahren wird nun so eingerichtet, dass sich für r_k nach der k-ten Näherung ein Ausdruck von der Form

$$r_{k+1}=F_{k+1}(A_0)b^0$$

ergibt. Dabei bedeuten

$$F_{k+1}(x)=\frac{1-T_{k+2}(x)}{2(k+2)^2x}=\frac{\sin^2(k+2)\frac{1}{2}\vartheta}{(k+2)^2\sin^2\frac{1}{2}\vartheta};$$

$$A_0=\frac{1}{\lambda_M}A;\quad b^0=\frac{1}{\lambda_M}b_0.$$

Den Rekursionsformeln der Tschebyscheffpolynome entsprechen hier Rekursionsformeln für die einzelnen Näherungen. Beide Näherungsverfahren werden nun miteinander kombiniert. Denkt man sich den Lösungsvektor nach Eigenvektoren entwickelt, so bewirkt Methode I, dass der den höheren Eigenwerten entsprechende Anteil des Restvektors rasch gegen Null konvergiert. Nach einer Vorbehandlung mit Methode I kommt man dann bei Methode II mit einer verhältnismässig geringen Anzahl von Schritten aus. Einer eingehenden Betrachtung werden auch insbesondere Systeme unterzogen, die nahe einem singulären System sind. Naturgemäss liefert das Verfahren auch eine Basis zur Ermittlung von Eigenwerten. *P. Funk.*

Slobodyanskiĭ, M. G. Estimates of the error of an approximate solution in linear problems reducing to variational ones, and their application to the determination of two-sided approximations in static problems of the theory of elasticity. Akad. Nauk SSSR. Prikl. Mat. Meh. 16, 449–464 (1952). (Russian)

Let A be a positive-definite, symmetric, linear operator defined on a linear manifold M of a real Hilbert space H (i.e., $(Au, u)\geq\gamma(u, u)$, and $(Au, v)=(u, Av)$, for u and v in M and some positive number γ, where (u, v) denotes the scalar product in H). In many linear boundary-value problems, the determination of the unknown function, or of its derivatives up to a certain order, or of a linear operator of the unknown function, may be shown to be equivalent to the determination of the scalar product (f, v), where $Au=f$, $Av=\psi$, with f and ψ given elements of H and u and v unknown elements of M. The present paper contains a method for the construction of numerical approximations to such a scalar product (f, v) which at the same time furnishes an estimate of the error. Concrete applications are made to boundary-value problems in two- and three-dimensional elasticity. (In the special case of the determination of the deflection of a clamped plate, the considerations are very closely related to those of J. B. Diaz and H. J. Greenberg [J. Math. Physics 27, 193–201 (1948); these Rev. 10, 213].) For the deflection $w(0, 0)$ at the center of a clamped square plate of side 2, the author obtains the following inequality: $|w(0, 0)-0.02030p/B|<0.00023p/B$, where B is the flexural rigidity and p is the uniform transverse load. There are also numerical applications to the bending moments and the deflection of a semicircular uniformly loaded clamped plate. *J. B. Diaz* (College Park, Md.).

Slobodyanskiĭ, M. G. Estimate of the error of the quantity sought for in the solution of linear problems by a variational method. Doklady Akad. Nauk SSSR (N.S.) 86, 243–246 (1952). (Russian)

The present note contains a method for the approximation of a scalar product (f, v) which at the same time furnishes an estimate of the error. (Here f is a given element of a real Hilbert space H, v is an unknown element of a linear manifold M of H, and $Au=f$, $Av=\psi$, where ψ is another given element of H, v is another unknown element of M, and A is a positive-definite, symmetric, linear operator defined on M with values in H.) Applications to boundary value problems in elasticity are sketched. A fuller account is contained in another paper [see the preceding review]. *J. B. Diaz* (College Park, Md.).

Jacobsen, L. S. On a general method of solving second-order ordinary differential equations by phase-plane displacements. J. Appl. Mech. 19, 543–553 (1952).

The author transforms the equation (1) $m\ddot{x}+G(x, \dot{x}, t)=0$ to the form $dx/dv=-v/(x+\delta)$ (the minus sign is missing in the printed paper) in which δ is a function of x, $\dot{x}$, and t, and from which a simple construction gives the slope in the vx-plane. The construction of short arcs gives values for changes in x, $\dot{x}$, and t and from these a solution of (1) is constructed step-by-step. The method is illustrated in considerable detail by worked examples. *W. E. Milne.*

Stiefel, E. Two applications of group characters to the solution of boundary-value problems. J. Research Nat. Bur. Standards 48, 424–427 (1952).

Soit un problème de conditions aux limites dont l'équation est $Df=0$ dans une région R. La méthode des différences permet de remplacer l'équation par un système aux différences $D_0f=0$ sur un treillis R_0. Soit G un groupe de transformations (symétries rotations qui transforme R_0 en lui-même et laisse D_0 invariant. L'emploi des caractères de G permet de décomposer le système D_0 en systèmes partiels indépendants. Deux applications sont données. *J. Kuntzmann* (Grenoble).

Fjørtoft, Ragnar. On a numerical method of integrating the barotropic vorticity equation. Tellus 4, 179–194 (1952).

Under the assumption of no divergence, no friction, no vertical motion, and negligible temperature advection, the vorticity equation becomes: $\nabla^2(\partial\psi/\partial t)=-\mathbf{V}\cdot\nabla(\nabla^2\psi)$, where ψ is the stream function, and $\mathbf{V}$ the horizontal velocity which can be determined from weather maps. This equation has been solved previously by laborious stepwise numerical integration involving a relaxation method for the solution for $\partial\psi/\partial t$. The present paper substitutes simple graphical approximate solutions of the equation. It makes use of the fact that the equation implies that isopleths of $\nabla^2\psi$ travel with the wind. Thus the equation is solved in two steps: 1) The isopleths of a numerical approximation of $\nabla^2\psi$ are moved with the speed of a smoothed wind, giving a forecast field of $\nabla^2\psi$. 2) ψ is found from $\nabla^2\psi$ by a simple graphical technique based on Fourier expansion of ψ as function of x

and y. Small corrections for the earth's curvature are also suggested. The total time of the integration for an area covering the North Atlantic and Europe is 2–3 hours for one man without special aids. The results of the method agree well with those obtained with electronic computers.
H. Panofsky (State College, Pa.).

Diaz, J. B., and Roberts, R. C. Upper and lower bounds for the numerical solution of the Dirichlet difference boundary value problem. J. Math. Physics **31**, 184–191 (1952).

Dans la solution d'un problème de Dirichlet aux différences, on peut utiliser des analogues de fonctions sur harmoniques ou sous harmoniques. Elles fournissent une borne supérieure et une borne inférieure de la solution. On les obtient par exemple par relaxation en s'arrangeant pour que les résidus soient tous positifs (négatifs). Lorsque l'on dispose d'une borne supérieure et d'une borne inférieure fournissant des résidus du même ordre de grandeur, on obtient une borne approximation de la solution en formant leur moyenne arithmétique. *J. Kuntzmann* (Grenoble).

Dingle, R. B. Some magnetic properties of metals. V. Magnetic behaviour of a cylindrical system of electrons for all magnetic fields. Proc. Roy. Soc. London. Ser. A. **216**, 118–142 (1953).

The Wentzel-Kramers-Brillouin method is used to solve the Schrödinger equation for an electron moving in a uniform magnetic field H, the boundary of the system being a cylinder with its axis lying along the direction of the field. It is found that there are two entirely different types of wave-function possible. *From the author's summary.*

Urban, P. Beitrag zum W.K.B.-Verfahren. Acta Physica Austriaca **6**, 181–194 (1952).

It is shown that the odd approximations of the W.K.B. method can be written as total differentials and that, with the exception of the first approximation, they therefore give no contribution to the quantum conditions. Asymptotic expansions for solutions of the Schrödinger equation can be found which do not show Stokes' phenomenon. The expansions are found by using Bessel functions rather than trigonometric functions as approximations to the solutions and then replacing the Bessel functions by their asymptotic expansions. *T. E. Hull* (Vancouver, B. C.).

Samuelson, Paul A. Rapidly converging solutions to integral equations. J. Math. Physics **31**, 276–286 (1953).

The author considers the Fredholm integral equation

$$E(x, z)=F(x, z)+U(x, z)+\int_a^b K(x, s)U(s, z)ds=0,$$

the corresponding operator equation being

$$F+U+K\cdot U=E(U, F, K)=0.$$

It is assumed that one and only one solution U exists. He proposes the iteration method

$$U_{n+1}=U_n-E(F, U_n, K)-j\cdot E(F, U_n, K)$$

with j as arbitrary integral operator. This generalizes the well known method of Neumann ($j=0$) and gives the exact solution $U_{n+1}=U$ for $j=k$, k denoting the operator of the reciprocal kernel of K. In order to ensure and to speed up convergence of the sequence U_n, j should be taken as approximation to k. Some other methods, hitherto proposed, are covered by the author's iteration. The case $j=\text{const.}\neq0$ presents the iteration as investigated by Wiarda and Bückner. The case $j(x, z)=j(x)=-\bar{K}(x)/(1+(b-a)\bar{\bar{K}})$ with $\bar{K}$ as mean of $K(x, z)$ over z and $\bar{\bar{K}}$ as mean of $K(x, z)$ over x and z gives the method of C. Wagner. Some other cases with a degenerate $j(x, z)$ are also discussed, and numerical examples are compared with each other. In a quite natural way, the author finds the iteration

$$k_{n+1}=k_n-(1+k_n)\cdot E(K, k_n, K)$$

for calculating the reciprocal kernel k. He notes that this is already known for inverting matrices. Application to integral equations, however, has already been recommended by Bodewig some years ago. [See Bückner, Die praktische Behandlung von Integralgleichungen, Springer, Berlin, 1952, §23; these Rev. **14**, 210.] *H. Bückner* (Berlin).

Yoshikawa, Hirosaku. Sur la représentation conforme d'un certain domaine. Mem. Fac. Engrg. Kyushu Univ. **12**, 223–227 (1950).

The Theodorsen-Garrick integral equation [NACA Rep. no. **452** (1933)] is derived. The result of the first iteration is used to approximate the resultant force acting on an airfoil in two-dimensional incompressible inviscid flow.
C. Saltzer (Cleveland, Ohio).

Yoshikawa, Hirosaku. Sur un problème de la convergence dans la représentation conforme. Mem. Fac. Engrg. Kyushu Univ. **12**, 229–234 (1951).

The author proposes modifying Koebe's "Schmiegungsverfahren" by a method similar to Ringleb's method [Nat. Res. Council Canada, Div. Mech. Eng., Tech. Translation no. **TT-70** (1948); these Rev. **11**, 341] and shows that for convex regions the modified method converges more rapidly.
C. Saltzer (Cleveland, Ohio).

Wang, Ming Chen, and Guth, Eugene. On the theory of multiple scattering, particularly of charged particles. Physical Rev. (2) **84**, 1092–1111 (1951).

The general theory of the elastic multiple scattering of particles with a strongly anisotropic scattering function is investigated without making the small-angle approximation. The rigorous transport equation is used and approximations are introduced at a later stage. The paper consists of four parts. In the first part the general formulation of the problem is given. The approximations involved in the existing theories of small-angle forward scattering are discussed in some detail. In the second part the spherical harmonic method is formulated in a manner so as to permit an explicit expression for the general nth approximation. There is an ambiguity both in (a) the way of defining successive approximations and in (b) the way of introducing approximate boundary conditions. We have chosen (b) to give the best approximation to the exact solution of the Schwarzschild-Milne problem. In the third part it is shown that our choice of (a) for the spherical harmonic method leads to the same final formulas as the gaussian quadrature method. The relation of these two methods is discussed in detail. In the fourth part the problem of anisotropic multiple scattering is reduced to a quasi-isotropic one by using a generalized Goudsmit-Saunderson type distribution function (defined also for back scattering) as a first approximation. (From the author's summary.) *J. M. Luttinger.*

Cruickshank, D. W. J. On the relations between Fourier and least-squares methods of structure determination. Acta Cryst. **5**, 511–518 (1952).

Neĭšuler, L. Ya. On conditions of single-valuedness of representations of functions of n variables by superpositions of n functions of two variables (i.e. n-term representations). Doklady Akad. Nauk SSSR (N.S.) 85, 1211–1214 (1952). (Russian)

The author treats the question of representing a function of n variables by means of n functions of two variables each. Representations of a special type are defined which turn out to be unique. Representations such as $f[g(x, y), g(x, h(y, z))]$ are excluded. [For previous papers on this subject see these Rev. 8, 171; 9, 104, 159, 208, 470, 735; 10, 149, 359, 741.]

D. H. Lehmer (Los Angeles, Calif.).

***Description of a magnetic drum calculator.** By the Staff of the Computation Laboratory. Harvard University Press, Cambridge, Mass., 1952. xii+318 pp. (1 plate). $8.00.

The book, one of a series put out by the Computation Laboratory of Harvard University, is concerned with the so-called Mark III Calculator, which was designed and constructed for the Bureau of Ordnance, United States Navy, by the staff of the Computation Laboratory. This machine was removed from Harvard to the Naval Proving Ground at Dahlgren, Virginia, in the spring of 1950.

The text itself should best be viewed as a quite technical description from an engineering point of view of the Mark III Calculator and is both thorough and compendious. The major interest in the work will lie not only with those engineers interested in computers but also with those applied mathematicians who are interested in programming problems for this particular machine. The text proceeds from an overall description of the calculator and of its basic circuits through a series of detailed descriptions of the principal logical organs of the machine to a discussion of problem preparation, coding, etc. for the machine. The book is well illustrated by a considerable number of photographs of important components and of circuit drawings to help clarify the reader on the details of the calculator. In the chapter on those functions viewed as elementary by the machine a quite detailed discussion is given of the approximation techniques used and in the chapter on problem preparation there are a number of examples coded to help the reader familiarize himself with the details of coding for this calculator.

While this book will not be of great interest to a large part of the mathematical community it is an important work in that it does make available in an immediate form a complete and detailed description of the Mark III Calculator to those who have occasion to utilize the machine.

H. H. Goldstine (Princeton, N. J.).

Platzman, George W. Some remarks on high-speed automatic computers and their use in meteorology. Tellus 4, 168–178 (1952).

Wilkes, M. V. The use of a 'floating address' system for orders in an automatic digital computer. Proc. Cambridge Philos. Soc. 49, 84–89 (1953).

Robinson, A. A. Multiplication in the Manchester University high-speed digital computor. Electronic Engrg. 25, 6–10 (1953).

Felker, J. H. Typical block diagrams for a transistor digital computer. Elec. Engrg. 71, 1103–1108 (1952).

Lode, Tenny. The realization of a universal decision element. J. Computing Systems 1, 14–22 (1952).

Speiser, Ambros P. Rechengeräte mit linearen Potentiometern. Z. Angew. Math. Physik 3, 449–460 (1952).

Jowett, G. H. A simply constructed adding machine. Math. Gaz. 36, 267–269 (1952).

Fritz, Norman L. Analog computers for coordinate transformation. Rev. Sci. Instruments 23, 667–671 (1952).

Tucker, M. J. A note on electronic analogue integration and differentiation. Electronic Engrg. 25, 35 (1953).

Symon, Keith R., and Poplawsky, Robert P. An electronic differential analyzer. Amer. J. Phys. 21, 53–61 (1953).

Saralegui, Antonio M. Accuracy and efficiency of stereoplotting instruments. Photogrammetric Engrg. 18, 901–932 (1952).

Vaughan, D. C. Relaxation methods. A three-dimensional mechanical analogy. Quart. J. Mech. Appl. Math. 5, 462–465 (1952).

Boscher, Jean. Sur la détermination numérique de fonctions biharmoniques par un procédé analogique de réseaux superposés. C. R. Acad. Sci. Paris 236, 44–46 (1953).

Alekseev, A. S. An electronic model of a two-position regulator of temperature with a lead zone. Doklady Akad. Nauk SSSR (N.S.) 87, 393–396 (1952). (Russian)

RELATIVITY

Abelé, Jean. La vitesse, grandeur qualitative, et la mécanique relativiste. C. R. Acad. Sci. Paris 235, 1007–1009 (1952).

On peut distinguer deux catégories de grandeurs: les grandeurs repérables et les grandeurs mesurables. Les premières ne supposent l'existence entre divers états d'une même grandeur, que d'une relation d'ordre (un exemple: le temps). Pour les grandeurs de la deuxième catégorie on exige en outre une opération concrète de composition exprimable par l'addition des nombres réels. Une grandeur intermédiaire entre ces deux catégories est nommée une grandeur qualitative si l'opération concrète de composition dont l'existence est supposée constitue un groupe isomorphe au groupe additif des nombres réels. Après cette définition la vitesse est une grandeur qualitative. La loi de décomposition s'identifie sous de certaines conditions à la loi relativiste de composition.

J. Haantjes (Leiden).

Malvaux, Pierre. Recherche d'une loi intrinsèque de composition des vitesses. C. R. Acad. Sci. Paris 235, 1009–1011 (1952).

La notion de dépassement suffit à définir la vitesse comme une grandeur repérable [voir l'analyse ci-dessus]. Si u, v, w sont les vitesses de M, A, M par rapport à A, B, B, il existe

une relation $w=f(u,v)$. Cette relation satisfait aux conditions formelles d'un groupe isomorphe à $W=U+V$. Alors il existe une fonction $F(x)$ telle que la loi de composition des vitesses $w=f(u,v)$ équivaut à $F(w)=F(u)+F(v)$. De l'axiome de l'existence d'une borne supérieure des vitesses ($c=1$) on trouve $F(x)=\int_0^x (K(x)/(1-x^2))dx$. Sous de certaines conditions on trouve $K(x)=1$, la loi relativiste de composition, d'où on déduit les formules de Lorentz.

J. Haantjes (Leiden).

Hlavatý, Václav. The elementary basic principles of the unified theory of relativity. B. J. Rational Mech. Anal. 2, 1–52 (1953).

This is the second of the author's projected series of three papers on the unified field-theory. [For paper A, see same J. 1, 539–562 (1952); these Rev. 14, 416.] The present paper, which is divided into three parts, is concerned with the existence of a connection $\Gamma^{\nu}{}_{\lambda\mu}$ such that

$$\text{(a)}\qquad \partial_\omega g_{\lambda\mu}=\Gamma^{\alpha}{}_{\lambda\omega}g_{\alpha\mu}+\Gamma^{\alpha}{}_{\omega\mu}g_{\lambda\alpha}$$

(note $\omega\mu$, not $\mu\omega$, in the final term) for given asymmetric fundamental tensor $g_{\lambda\mu}$. In Part I it is shown that these equations for the Γ's may have no solutions, or a single solution, or more than one, according as certain conditions are satisfied. Confining himself ultimately to the case when there is exactly one solution, the author obtains it in the form

$$\text{(b)}\qquad \Gamma^{\nu}{}_{\lambda\mu}=\{{}^{\nu}_{\lambda\mu}\}+\tfrac{1}{2}(\nabla_{(\alpha}g_{\gamma)\beta}+\nabla_{\beta}g_{\alpha\gamma})+y^{\alpha\beta\gamma}_{\xi\nu\epsilon}[h^{\xi\nu}\delta^{\epsilon\kappa}_{\lambda\mu}+2h^{\xi\nu}\delta_{(\lambda}{}^{\epsilon}k_{\mu)}{}^{\xi}],$$

where the Christoffel symbols are formed from the symmetric part $h_{\lambda\mu}$ of $g_{\lambda\mu}$, ∇_α denotes covariant differentiation with respect to those symbols, $k_{\lambda\mu}$ is the skew-symmetric part of $g_{\lambda\mu}$, and $y^{\alpha\beta\gamma}_{\xi\nu\epsilon}$ is a certain very complicated tensor depending upon the $g_{\lambda\mu}$. Part II begins with a consideration of the case $n=4$, comparative simplicity being lent to the analysis by the use of a non-holonomic system of reference. A return is then made to the case of any n, because, although equations (b) provide a theoretically complete answer to the problem of solving equations (a), the physical application of the formula involves very heavy computations. Special consideration is therefore given to the cases, each based on a special structure of the tensor $g_{\lambda\mu}$, in which these computations may be avoided. In particular, the cases are treated in which the autoparallels defined by $\Gamma^{\nu}{}_{\lambda\mu}$ are geodesics associated with $\{{}^{\nu}_{\lambda\mu}\}$, that is, when (in case $n=4$ and the signature of $h_{\lambda\mu}$ is $---+$) they are the world-lines of particles or of light-pulses in classical relativity. In Part III a method is developed for obtaining, in a finite number m of steps, an approximate solution ${}^{m}\Gamma^{\nu}{}_{\lambda\mu}$ of equations (a) such that the difference $\Gamma^{\nu}{}_{\lambda\mu}-{}^{m}\Gamma^{\nu}{}_{\lambda\mu}$ is as small as is desired, the method being based upon the assumption that $k_{\lambda\mu}$ is small compared with $h_{\lambda\mu}$. The paper concludes with a short discussion of the consequence of taking, instead of equations (a), the same equations with the subscripts $\omega\mu$ in the final term reversed. It turns out that the equations thus amended, unlike equations (a) themselves, impose a condition upon $g_{\lambda\mu}$, namely $\det|g_{\lambda\mu}|/\det|h_{\lambda\mu}|=\text{const}$.

H. S. Ruse.

Hlavatý, Václav. The Schrödinger final affine field laws. Proc. Nat. Acad. Sci. U. S. A. 38, 1052–1058 (1952).

Schrödinger [Proc. Roy. Irish Acad. Sect. A. 51, 163–171, 205–216; 52, 1–9 (1948); these Rev. 9, 310, 311] bases his field laws upon an affine connection $\Gamma^{\nu}{}_{\lambda\mu}$ satisfying $\Gamma^{\alpha}{}_{\lambda\alpha}=\Gamma^{\alpha}{}_{\alpha\lambda}$ and

$$\text{(i)}\qquad \partial_\omega g_{\lambda\mu}=\Gamma^{\alpha}{}_{\lambda\omega}g_{\alpha\mu}+\Gamma^{\alpha}{}_{\omega\mu}g_{\lambda\alpha},$$

where (ii) $g_{\lambda\mu}=P_{\lambda\mu}+F_{\lambda\mu}$. Here $P_{\lambda\mu}$ is the Ricci tensor formed from $\Gamma^{\nu}{}_{\lambda\mu}$ and $F_{\lambda\mu}=\tfrac{2}{3}(\partial_\lambda F_\mu-\partial_\mu F_\lambda)$, F_λ being a vector. Substitution from (ii) in (i) gives a system of 68 partial differential equations for 68 unknowns $\Gamma^{\nu}{}_{\lambda\mu}$, F_μ. The author's object is to reduce this system to another system of 20 equations for the 20 unknowns $g_{\lambda\mu}$, F_μ and to find $\Gamma^{\nu}{}_{\lambda\mu}$ explicitly in terms of $g_{\lambda\mu}$ and F_μ. In fact he obtains the corresponding result for any n, and then seeks approximate solutions for the case $n=4$ on the assumption that the skew part $k_{\lambda\mu}$ of $g_{\lambda\mu}$ is small. He obtains the following approximate results: The symmetric part $h_{\lambda\mu}$ of $g_{\lambda\mu}$ satisfies $h_{\lambda\mu}=H_{\lambda\mu}$, where $H_{\lambda\mu}$ is the Ricci tensor formed from $h_{\lambda\mu}$. As a solution of this he takes $h_{\lambda\mu}$ as the metric tensor of a Riemannian space of constant curvature 4. The vector F_μ is a gradient, so that $F_{\lambda\mu}=0$, and the coefficients of connection are given by

$$\Gamma^{\nu}{}_{\lambda\mu}=\{{}^{\nu}_{\lambda\mu}\}+\tfrac{1}{2}(\nabla_\lambda k^{\nu}{}_{\mu}+\nabla_\mu k^{\nu}{}_{\lambda}+\nabla^{\nu}k_{\lambda\mu}),$$

where the Christoffel symbols are formed from the $h_{\lambda\mu}$ and ∇_λ denotes covariant differentiation with respect to them. The bivector-density $f^{\lambda\mu}$ dual to the bivector $k_{\lambda\mu}$ satisfies a Maxwellian system of equations

$$\nabla_\omega f^{\omega\nu}=\rho u^{\nu}.$$

H. S. Ruse (Leeds).

Cap, Ferdinand. Über allgemeine Relativitätstheorie und einheitliche Feldtheorie. Acta Physica Austriaca 6, 135–156 (1952).

The paper contains summaries of the main features of special relativity, general relativity, and the unified field-theory (gravitation and electromagnetism) of Schroedinger.

G. C. McVittie (Urbana, Ill.).

Ikeda, Mineo. A note on some special spherically symmetric space-times. Tensor (N.S.) 2, 102–107 (1952).

Let K_{ijlm} denote the curvature tensor and K_{ij} the Ricci tensor in a spherically symmetric space-time, that is, one for which ds^2 is reducible to the form

$$ds^2=-A(r,t)dr^2-B(r,t)(d\theta^2+\sin^2\theta d\phi^2)+C(r,t)dt^2,$$

and let ∇_h denote covariant differentiation. Following Takeno [J. Math. Soc. Japan 3, 317–329 (1951); these Rev. 13, 985], the author finds the most general spherically symmetric space-times in a number of special cases. Solutions for the separate cases in which $K_{ij}=0$, $K_{ij}=\alpha g_{ij}$, $\nabla_h K_{ijlm}=0$, $\nabla_h K_{ij}=0$ being already known, he considers the seven cases in which, respectively,

$$\nabla_h K_{ij}{}^{lm}=\phi_h K_{ij}{}^{lm},\quad \nabla_h K_i{}^j=\phi_h K_i{}^j,$$
$$\nabla_h K_{ij}{}^{lm}=\phi_h(\delta_i{}^l\delta_j{}^m-\delta_i{}^m\delta_j{}^l),\quad \nabla_h K_i{}^j=\phi_h\delta_i{}^j,$$
$$\nabla_h K_i{}^j=\tfrac{1}{4}(\nabla_h K_s{}^s)\delta_i{}^j,\quad \nabla_s K_{ij}{}^{ls}=0,\quad \nabla_s K_i{}^s=0.$$

In the first of these, with $\phi_h\neq 0$, there are no non-flat solutions, but in most cases non-trivial solutions do exist.

H. S. Ruse (Leeds).

Jordan, P. Die Nichtgültigkeit des Birkhoffschen Satzes in der erweiterten Gravitationstheorie. Z. Physik 133, 558–560 (1952).

Birkhoff's theorem in general relativity states that the only spherically symmetric statical solution of Einstein's equations (with the cosmical constant equal to zero) for empty space is the Schwarzschild solution. The author shows that this theorem is no longer true in his modified version of general relativity in which the constant of gravitation is regarded as a field variable.

G. C. McVittie.

Jánossy, L. On the physical interpretation of the Lorentz transformation. Acta Phys. Acad. Sci. Hungar. 1, 391–422 (1952). (Russian summary)

An exhaustive effort to interpret the content of the special theory of relativity in terms of physical deformations of rods and clocks in motion with respect to a preferred local frame of reference in which the velocity of light is isotropic. It is held desirable to consider the fundamental transformations between moving observers to be Galilean, from which the Lorentz transformations result on applying the "Lorentz deformations": 1) FitzGerald contraction, 2) Lorentz slowing down of moving clocks, and 3) phase difference in moving clocks at different points. *H. P. Robertson.*

Risco, M. Interprétation d'un phénomène interférentiel par des observateurs en mouvement relatif. J. Phys. Radium (8) 13, 441–444 (1952).

By a detailed explicit argument the author shows that, given two systems of plane light waves with a common scalar amplitude, but with different frequencies and directions of propagation, it is possible to find a Lorentz transformation which reduces them to a system of standing waves (frequencies equal, directions of propagation opposed). The standing waves give a set of stationary interference fringes. He proposes to associate with these fringes a particle at rest relative to them with proper mass m_0 satisfying $m_0c^2=h\nu_0'$, where ν_0' is the frequency of the standing waves. In a further calculation, based on the conservation of momentum and energy for the transition from two photons to a particle, he obtains a proper mass twice as great; he avoids an inconsistency by supposing that two particles are formed. [The essentials of the argument can be presented very briefly in terms of 4-vectors, with use of Minkowskian coordinates. If f_r and f_r' are the frequency 4-vectors of the photons, a time-axis pointing in the direction of (f_r+f_r') gives standing waves. If the two photons yield a single particle, then by the law of conservation the 4-momentum of this particle is $(h/c^2)(f_r+f_r')$, and hence its proper mass is $2h\nu_0'/c^2$, which is the author's second result.] *J. L. Synge* (Dublin).

Ueno, Yoshio, and Takeno, Hyôitirô. On equivalent observers. Progress Theoret. Physics 8, 291–301 (1952).

The notion behind this program is that the transformations between the space-time coordinates of equivalent observers M, M' constitute a group $\mathfrak{G}$, and that the physical laws of interest are to be form-invariant under this group. The present investigation divides the problem of determining the group $\mathfrak{G}$ into two parts: a) the specification of the group $\mathfrak{G}_0$ of transformations between coordinate frames which may be used by a given observer M; and b) the determination of a larger group $\mathfrak{G}$, of which $\mathfrak{G}_0$ is a subgroup, defining the transformations between equivalent observers M, M' which are in relative motion. Such groups $\mathfrak{G}$ are obtained which contain as reference-frame subgroups: i) the 7-parameter group $\mathfrak{G}_0$ of all euclidean motions plus translation in time, and ii) the 3-parameter group $\mathfrak{G}_0'$ of euclidean rotations about a point. *H. P. Robertson.*

MECHANICS

Yamada, Kaneo. Fundamental theory of toothed gearing. I, II, III, IV, V, VI, VII. Proc. Japan Acad. 25, 84–89, 90–96, 97–102, 133–138, 139–144, 145–150, no. 10, 1–6 (1949).

The relative motion of two gears is obtainable by pure rolling (without sliding) of a curve on one gear over a curve on the other gear; the curves are called the pitch curves. The tooth profiles carried by the gears also roll on each other, but with sliding. In these papers, the well-known fundamental theorems of gearing are derived. They include the relations between the profile curves and the pitch curves, the derivation of the profile curves as roulettes of the pitch curves, the equations of the curves and the radii of curvature. The general theorems are applied to the two most prevalent systems of gearing, the cycloidal system and the involute system, to determine the equations of the profile curves and the paths of contact.

The plane theory is extended to curves on the surface of a sphere where the great circles play the role of straight lines. The general geometrical theory is developed for the rolling of curves over curves. Analytical expressions in terms of spherical polar coordinates are derived. The spherical analog of the Euler-Savary formula for plane rolling curves is derived and used. Applications are made to the determination of the spherical equations for the profile curves, the paths of contact between cycloidal bevel gears (small circles), between involute bevel gears (great circles), and between octoid bevel gears (conchoid or figure-eight curves). Expressions for the specific slidings are derived for each case.

The work is carried out with much greater mathematical detail than is found in textbooks. However, the thoroughness of the derivations is marred by the complete lack of figures which would make these papers excellent textbook material. *M. Goldberg* (Washington, D. C.).

Hughes, J. V. Possible motions of a sphere suspended on a string (the simple pendulum). Amer. J. Phys. 21, 47–50 (1953).

Supino, Giulio. Sopra i teoremi di Lord Rayleigh. Boll. Un. Mat. Ital. (3) 7, 261–266 (1952).

Consider a conservative mechanical system of n degrees of freedom characterized by the Lagrangian

$$L=\sum a_{ik}q_iq_k+\sum b_{ik}\dot{q}_i\dot{q}_k,$$

where the a_{ik}, b_{ik} are constants. The author shows that for a special, but fairly large, class of such systems the natural frequencies are independent of the masses involved. The proofs are based on the extremal properties of the eigenvalues and on dimensional considerations. *W. Wasow.*

Nadile, Antonio. Traiettorie dinamiche di un sistema anolonomo e famiglie naturali di curve anolonome. Rivista Mat. Univ. Parma 3, 115–129 (1952).

In the first part of this paper the author gives the differential equations of motion of a non-dissipative dynamical system, subject to non-holonomic constraints, in an intrinsic form. The second part of the paper is devoted to a discussion of the differential equations defining the extremals of a variational problem of the form

$$\delta\int_A^B F(q_1,\cdots,q_n,q_1',\cdots,q_n')ds=0,$$

where the variations of the q's are subject to a set of non-

holonomic constraints of the form

$$\sum_{j=1}^{n} b_{ij}(q_1, \cdots, q_n)\delta q_j = 0 \quad (i=1, \cdots, m).$$

Various special cases, in which the problem admits of more or less realistic interpretations in terms of dynamics, are considered. The third part of the paper gives a brief discussion of a few of the properties of the families of curves arising in the preceding problems. *L. A. MacColl.*

Gallissot, François. Sur l'origine des impossibilités et des indéterminations dues aux liaisons. C. R. Acad. Sci. Paris **235**, 937–939 (1952).

Une liaison $a^h(p, q, t)=0$ est dite de classe U si en supprimant cette liaison à l'instant t_0 le mouvement qui se produit est tel que a^h garde un signe constant dans le voisinage $t_0<t<t+\epsilon$. L'auteur obtient le résultat suivant: Le système S' (p liaisons de classe U) peut, dans le voisinage de t_0, présenter 2^p éventualités. Les conditions initiales ne déterminent un mouvement unique que si tous les mineurs diagonaux d'une certaine matrice sont positifs.
J. Haantjes (Leiden).

Pailloux, Henri. Principes et liaisons en mécanique rationnelle. Ann. Univ. Saraviensis **1**, 201–210 (1952).

An exposition of the principles of classical mechanics: Newton's second law of motion, virtual work, a generalized Hamiltonian principle, Lagrange's equations for holonomic systems and for various types of non-holonomic systems, and Appell's equations. *D. C. Lewis* (Baltimore, Md.).

Pailloux, Henri. Une transformation des équations de Lagrange. Ann. Univ. Saraviensis **1**, 217–219 (1952).

Equations of Lagrange written in terms of what some authors call "quasi-coordinates," amounting to the introduction of arbitrary linear combinations of the velocity components as new unknowns in place of the velocity components themselves. *D. C. Lewis* (Baltimore, Md.).

Merkulov, V. I. On a problem of Žukovskiĭ. Akad. Nauk SSSR. Prikl. Mat. Meh. **16**, 633–634 (1952). (Russian)

N. E. Žukovskiĭ [Selected papers, vol. II, Moscow-Leningrad, 1948, pp. 152–309; these Rev. 11, 574] investigated the problem of motion of a solid containing cavities filled with liquid in motion. The paper under review is concerned with a particular case of the problem when the solid is axially symmetric and has a doubly-connected cavity of revolution around the axis of symmetry. The problem can be reduced to that of motion about a fixed point O lying on the axis of symmetry. Assume that gravity is the only exterior force acting. Take the axis of symmetry as the z-axis of a moving coordinate system $Oxyz$, and denote by C and A the polar and the equatorial moments of inertia of the solid respectively. Let p, q, r be the components along the axes $Oxyz$ of the angular velocity of the solid, and let m be the mass of the liquid. The order of connectivity of the cavity being two, a family of reconcilable irreducible circuits can be drawn in it. Let the circulation in any of these circuits be k. Žukovskiĭ showed that for any ring-shaped cavity of revolution the momentum of the liquid, relative to the solid, is $mk/2\pi$ and is directed along the axis of revolution. Knowing the momentum components Ap, Aq, $Cr+mk/2\pi$ of the whole system, the momentum theorem is applied. Denote by α, β, γ the direction cosines of the z-axis with respect to a fixed coordinate system $OXYZ$, and consider the case of motion with small angle of nutation, i.e., assume that $\gamma=1$ and $\alpha, \beta, \alpha', \beta'$ are small quantities of the first order. Neglecting third-order quantities and introducing a new variable $\psi=\alpha+i\beta$, it follows from the momentum theorem that ψ satisfies the equation

$$(1) \qquad A\psi''-i(Cr+mk/2\pi)\psi'-Pa\psi=0,$$

where P is the weight of the system under consideration and a is the distance between the fixed point O and the center of mass.

The characteristic equation of (1) will have pure imaginary roots if and only if

$$(Cr/2A+mk/4\pi A)^2 \geq Pa/A.$$

This inequality determines in the (k, r)-plane two parallel straight lines, the region between which is the strip of instability of motion. The points of intersection of these lines with the k-axis are $\pm 4\pi(PAa)^{1/2}/m$, which correspond to the value of circulation k which ensures the stability of the equilibrium in the vertical position of the vessel, the vessel being considered as at rest. *E. Leimanis.*

Hydrodynamics, Aerodynamics, Acoustics

Popov, S. G. Remark on the integrals of Bernoulli and Lagrange (Cauchy). Moskov. Gos. Univ. Učenye Zapiski **152**, Mehanika 3, 43–46 (1951). (Russian)

The author points out that in barotropic flow of an ideal fluid subject to conservative extraneous forces the usual Bernoullian expression B satisfies $dB=d\mathbf{x}\cdot\mathbf{v}\times\mathbf{w}$, where $d\mathbf{x}$ is the element of arc, $\mathbf{v}$ the velocity, and $\mathbf{w}$ the vorticity, and hence $B=$const. on any curve such that $d\mathbf{x}\cdot\mathbf{v}\times\mathbf{w}=0$. The usual statements apply to either (a) any curve lying on a surface containing both vortex-lines and stream-lines, or (b) irrotational or Gromeko-Beltrami motion. The author notices that certain mixed conditions, involving both the curve and the motion, are also sufficient. His simplest example is a motion such that both velocity and vorticity are everywhere normal to a fixed direction: then any plane curve whose plane is normal to this direction satisfies the condition. In a frame rotating at angular velocity ω, the author obtains a formula of the type $d(B+B')=\cdots$, where B' is the centrifugal potential, and he discusses some analogous results which are corollaries of it.

Since the author makes some point of his attributions of names, the reviewer adds the following notes. (1) The result of Daniel and John I. Bernoulli was derived by them subject only to the hydraulic approximation (1730, 1740). (2) What are now usually called the "Bernoulli equation for (possibly unsteady) barotropic irrotational flow" and the "Bernoulli equation for the stream-lines" were first obtained by Euler, Mém. Acad. Roy. Sci. Berlin **1755**, 274–315, 316–361 (1757); see §XXVIII of the former and §§LI–LII of the latter. (The former in the unsteady case is called the "Lagrange (Cauchy) equation" by the author; the latter is often attributed to Lagrange in the French literature.) (3) The extension of the latter to surfaces tangent to both stream- and vortex-lines, which the author does not mention, is due to Lamb [Proc. London Math. Soc. (1) **9**, 91–92 (1878)]. (4) What the author calls the "Gromeko-Lamb equations" were derived by Lagrange in 1781 [Oeuvres, t. 4, Gauthier-Villars, Paris, 1869, pp. 695–748, see §14]. *C. Truesdell.*

Ertel, Hans. Ein Theorem über asynchron-periodische Wirbelbewegungen kompressibler Flüssigkeiten. Miscellanea Academica Berolinensia, vol. I, pp. 62–68. Akademie-Verlag, Berlin, 1950.

The author considers steady barotropic motions of an ideal fluid subject to conservative extraneous force, under the special assumption that all streamlines are closed curves. The period of rotation is then $\tau=\tau(a, b, c)$, where a, b, c are the material coordinates. Let $H(a, b, c)$ be the usual Bernoullian expression, and put $L=H+v^2$. Put $W=\int_0^\tau L dt$, where the integration is carried out for a fixed particle. By integrating the material equations of motion over one period, the author derives the theorem $\tau=dS/dH$, where $S=W+H\tau$. This is the analogue of a proposition in the Hamilton-Jacobi theory. The author shows that S is in fact the circulation around the closed streamline, and thus, since this streamline must lie upon a surface $H=$const., the theorem can be interpreted in kinematical terms. *C. Truesdell.*

Ericksen, Jerald L. Thin liquid jets. J. Rational Mech. Anal. 1, 521–538 (1952).

The author sets out to generalize certain results of Boussinesq [Mém. Acad. Sci. Inst. France **23**, 1–680 (1877), pp. 639–659] by considering a stationary liquid jet of small but variable thickness (water bell) in a medium of pressure p_0. If p is the pressure in the jet at the boundary, the surface tension condition $p=p_0-AC$ is assumed, where A is a positive constant and C is the mean curvature at the boundary. The further assumption is made that the jet has a "midway" surface Σ_1 on which the pressure is also p_0. He then proceeds to set up equations of motion and to describe Σ_1 for bells which are not necessarily axisymmetric. He derives a rather limited class of solutions for the case of constant body force and a much larger class for the case of zero body force. In this latter case it is shown that the equations of motion are invariant when the thickness of the jet is multiplied by ϕ^2 and the velocity vector by $1/\phi$, where ϕ is a function of position, constant along a streamline. This substitution principle enables an infinite number of solutions to be derived when one is known.
L. M. Milne-Thomson (Greenwich).

Slezkin, N. A. Plane flow of an ideal fluid about a gas-filled shell. Moskov. Gos. Univ. Učenye Zapiski **152**, Mehanika **3**, 61–75 (1951). (Russian)

The non-rigid gas-filled shell is assumed to be convex and to be under constant tension. External forces are not treated. The boundary condition on the surface of the body becomes $a+bV^2=r_1^{-1}+r_2^{-1}$ where r_1 and r_2 are the principal radii of curvature, a and b are constants, and V is the velocity. The two-dimensional case is treated in detail. In this case the problem may be reduced to the solution of the following integral equation:

$$T(\beta)=-\lambda\int_0^\pi [e^{T(\alpha)}\sin^2\alpha+ke^{-T(\alpha)}]\ln[4\sin^2(\alpha-\beta)]d\alpha.$$

A process of successive approximations is introduced and proved convergent to a solution if the non-dimensional parameter λ is below a given bound (giving a relation between the arc length, the velocity at infinity, the tension, and the density). An approximate computation is carried out, giving an elliptical cross-section with the long axis perpendicular to the flow. *J. V. Wehausen.*

Gerber, Robert. Un théorème d'unicité pour les écoulements d'un liquide parfait, pesant. C. R. Acad. Sci. Paris **235**, 693–694 (1952).

Continuing his earlier work on gravity waves [same C. R. **233**, 1261–1263 (1951); these Rev. **13**, 594], the author now announces conditions under which the problem considered in the cited paper has a unique solution.
J. V. Wehausen (Providence, R. I.).

Gerber, Robert. Sur l'existence des écoulements plans, permanents, irrotationnels, uniformes à l'infini, des liquides incompressibles. C. R. Acad. Sci. Paris **235**, 1601–1602 (1952).

The author considers the following problem in two-dimensional gravity waves in an inviscid fluid. Let the bottom be given by a curve S with horizontal asymptotes at both $+\infty$ and $-\infty$ and with non-positive bounded slope everywhere. Let the free surface be given by the (unknown) curve L. The domain A of the z-plane bounded by S and L is to be mapped by a univalent function onto the domain D of the $f(=\varphi+i\psi)$-plane given by $0\leqq\psi\leqq\psi_0$ so that the points at infinity correspond and so that the following boundary conditions are satisfied (where $df/dz=V_0e^{-i\omega(f)}$, $\omega=U+iT$): $\omega(-\infty+i\psi)=0$, $\omega(+\infty+i\psi)=k$, a constant to be determined, and $e^{3T}dT/d\varphi+gV_0^{-3}\sin U=0$ and $U\leqq 0$ for $\psi=\psi_0$. The author announces a proof of the existence of a solution under certain restrictions on the slope of S and the Froude number $g\psi_0/V_0^3$. The method, as sketched, is similar to that of an earlier note [cited in the preceding review].
J. V. Wehausen (Providence, R. I.).

Sretenskiĭ, L. N. Survey of works on the theory of waves during the time from 1917 to 1949. Moskov. Gos. Univ. Učenye Zapiski **152**, Mehanika **3**, 76–98 (1951). (Russian)

The title is somewhat misleading in that only work done in the USSR is considered [for a brief survey of non-Russian work see Stoker, Proc. Internat. Congr. Math., Cambridge, Mass., 1950, vol. 2, pp. 304–307; these Rev. **13**, 396]. The exposition follows closely that in the author's paper in "Mehanika v SSSR za tridcat' let" [pp. 279–299, Moscow-Leningrad, 1950; these Rev. **12**, 660] and takes up successively waves of finite amplitude, infinitesimal waves, and tidal waves; an extensive bibliography is included. A surprising omission is any discussion (or even listing of the paper) of Lavrent'ev's paper proving the existence of a solitary wave [Akad. Nauk Ukrain. RSR. Zbirnik Prac' Inst. Mat. **1946**, no. 8, 13–69 (1947); these Rev. **14**, 102].
J. V. Wehausen (Providence, R. I.).

Sretensky, L. N. Motion of a cylinder under the surface of a heavy fluid. NACA Tech. Memo. no. **1335**, 37 pp. (1952).

Translated from Trudy Central. Aero-Gidrodinam. Inst., no. **346** (1938).

Kochin, N. E. Steady vibrations of wing of circular plan form. NACA Memo. no. **1324**, 1–50 (1953).

Translated from Akad. Nauk SSSR. Prikl. Mat. Meh. **6**, 287–316 (1942); these Rev. **4**, 228.

Kochin, N. E. Theory of wing of circular plan form. NACA Tech. Memo. no. **1324**, 51–93 (1953).

Translated from Akad. Nauk SSSR. Prikl. Mat. Meh. **4**, 3–32 (1940).

Andersson, Bengt. On the stress-tensor of viscous isotropic fluids. Ark. Fys. 4, 501–503 (1952).

By an example the author shows that in a fluid in which the viscous stresses are functions of the rate of deformation only, a linear stress-deformation law in Couette or Poiseuille flow does not necessarily imply a linear relation in general.

C. Truesdell (Bloomington, Ind.).

Heinrich, G. Der Energietransport in strömenden Medien. Z. Angew. Math. Mech. 32, 286–288 (1952).

For a viscous compressible fluid, the author calculates the rate of change of flow of mechanical energy through an arbitrary surface element. [Since only part of the energy change can be accounted for by surface integrals, any such result is arbitrary. More general analysis has been given by the reviewer, Physical Rev. (2) 73, 513–515 (1948); these Rev. 9, 474].

C. Truesdell (Bloomington, Ind.).

Hamel, Georg. Spiral motions of viscous fluids. NACA Tech. Memo. no. 1342, 44 pp. (1953).

Translated from Jber. Deutsch. Math. Verein. 25, 34–60 (1916).

Ackeret, Jakob. Über exakte Lösungen der Stokes-Navier-Gleichungen inkompressibler Flüssigkeiten bei veränderten Grenzbedingungen. Z. Angew. Math. Physik 3, 259–271 (1952).

Comme il est bien connu les mouvements potentiels conviennent à un fluide visqueux incompressible; toutefois si dans un tel mouvement le fluide est en contact avec une paroi solide fixe la condition d'adhérence entraîne le repos. Mais on peut imaginer une paroi constituée par une membrane déformable qui se meut le long d'une surface fixe S de telle sorte qu'en chaque point la vitesse de la paroi soit justement égale à celle du fluide en mouvement potentiel autour de S; on obtient ainsi une solution exacte des équations de Navier-Stokes avec la condition d'adhérence réalisée. G. Hamel a consacré un mémoire à ces mouvements [Z. Angew. Math. Mech. 21, 129–139 (1941); ces Rev. 3, 92]. L'auteur du présent article retrouve certains résultats de Hamel et étudie des exemples nouveaux dont l'un est l'écoulement de Riabouchinsky et Demtchenko autour de deux plaques planes parallèles reliées par des lignes de discontinuité. Un autre exemple nouveau est l'écoulement autour d'une sphère. Dans tous ces mouvements on peut définir une "résistance", cette dernière étant le quotient par la vitesse au loin de la puissance dépensée pour maintenir la paroi en mouvement. Pour le cas de la sphère l'auteur obtient une "résistance" égale exactement au double de la résistance ordinaire en mouvement lent donnée par la formule de Stokes. Pour de grands nombres de Reynolds cette "résistance" est bien inférieure à la résistance ordinaire avec paroi fixe.

R. Berker (Istanbul).

van Dyke, Milton D. Impulsive motion of an infinite plate in a viscous compressible fluid. Z. Angew. Math. Physik 3, 343–353 (1952).

L'auteur étudie le mouvement d'un fluide visqueux et compressible, dû à la translation uniforme d'un plan indéfini initialement au repos. La méthode de la couche limite, déjà utilisée dans ce problème par Illingworth et Stewartson, est développée. Les équations simplifiées de la théorie de la couche limite donnent une première approximation du mouvement dans cette couche; dans la région du fluide extérieur la première approximation est fournie par les équations indéfinies de l'acoustique. Il est alors possible de construire de nouvelles approximations du mouvement dans les deux régions du fluide, en utilisant chaque fois des conditions aux limites plus précises sur la région de séparation. L'auteur développe ainsi le calcul de ces approximations jusqu'au troisième ordre. Celles-ci pourraient se poursuivre indéfiniment et elles donneraient alors une expression asymptotique de la solution pour des temps très grands.

R. Gerber (Toulon).

Mager, Artur. Generalization of boundary-layer momentum-integral equations to three-dimensional flows including those of rotating system. NACA Rep. no. 1067, ii+16 pp. (1952).

"Supersedes NACA Tech. Note no. 2310 (1951)"; these Rev. 12, 871.

Morawetz, Cathleen S. The eigenvalues of some stability problems involving viscosity. J. Rational Mech. Anal. 1, 579–603 (1952).

Soit un mouvement laminaire plan d'un fluide visqueux incompressible auquel on superpose des perturbations infiniment petites, également planes et dérivant d'une fonction de courant de la forme $\phi(y)e^{i\alpha(x-ct)}$. L'auteur étudie pour les trois problèmes qui seront indiqués ci-dessous les valeurs propres c et leur relation avec les valeurs propres du problème correspondant en fluide parfait. Les trois problèmes en question sont les suivants: 1) écoulement entre deux parois parallèles animées de vitesses différentes; 2) écoulement avec profil de vitesse symétrique entre deux parois parallèles fixes; 3) écoulement dans la couche limite le long d'une plaque plane. Sous certaines hypothèses générales l'auteur détermine une relation asymptotique entre c, α et αR (R désigne le nombre de Reynolds) pour les grandes valeurs de αR. Représentons par y_1 et y_2 les ordonnées des parois dans le problème 1, les ordonnées d'une paroi et de la droite médiane de l'écoulement dans le problème 2 et par y_1 l'ordonnée de la plaque dans le problème 3; l'auteur obtient le résultat suivant: $w=w(y)$ étant le profil de vitesses du mouvement laminaire non perturbé, exception faite de voisinages infiniment petits de $c=w(y_1)$ et $c=w(y_2)$, il n'existe pas en fluide visqueux d'autres perturbations instables ou neutres que celles correspondant aux perturbations instables ou neutres en fluide parfait; la méthode utilisée ne s'applique pas aux voisinages exclus.

R. Berker (Istanbul).

Lin, C. C. On Taylor's hypothesis and the acceleration terms in the Navier-Stokes equation. Quart. Appl. Math. 10, 295–306 (1953).

Pour établir une relation entre le spectre temporel et les corrélations spatiales de la turbulence en soufflerie, Taylor [Proc. Roy. Soc. London. Ser. A. 164, 476–490 (1938)] avait utilisé une hypothèse sur l'entraînement des tourbillons par le mouvement d'ensemble. L'auteur se propose de justifier l'hypothèse de Taylor dans le cas des écoulements homogènes à faible turbulence, et de la discuter dans le cas des écoulements non uniformes (couche limite). Dans la première partie de son travail, il considère les "termes d'acceleration"

$$a_i=\frac{\partial u_i}{\partial t},\quad \alpha_i=u_j\frac{\partial u_i}{\partial x_j},\quad \pi_i=-\frac{1}{\rho}\frac{\partial P}{\partial x_i},\quad \mu_i=\nu\Delta u_i,$$

qui composent les équations du mouvement et, calcule leurs diverses corrélations en deux points de l'espace, en fonction des corrélations doubles, triples, et quadruples, ces dernières

étant exprimées en fonction des premières grâce à une hypothèse, non rigoureusement justifiable, précédemment introduite par Batchelor [Proc. Cambridge Philos. Soc. 47, 359–374 (1951); ces Rev. 12, 874]. Il en déduit des inégalités entre ces diverses corrélations. En particulier, dans un écoulement homogène de vitesse uniforme U, il trouve que $\langle(\partial u/\partial t)^2\rangle/U^2\langle(\partial u/\partial x)^2\rangle$ est de l'ordre de 5 fois le carré de l'intensité de la turbulence, ce qui justifie l'hypothèse de Taylor. Si U varie, l'hypothèse de Taylor n'est pas justifiable par des raisonnements a priori. Dans la couche limite, sa validité dépend de la longueur d'onde des tourbillons, comparée à l'épaisseur de la couche limite. *J. Bass.*

★**A selection of tables for use in calculations of compressible airflow.** Prepared on behalf of the Aeronautical Research Council by the Compressible Flow Tables Panel. Oxford, at the Clarendon Press, 1952. viii+143 pp. $8.00.

This book contains a collection of tables of the most commonly used functions occurring in the calculation of the steady flow of air for which the ratio of specific heats is 1.4. Part I gives relations of quantities in steady isentropic flow and subdivides into five tables: in Tables I.1 and I.2, local Mach number M is taken as independent variable and covers the ranges 0(0.1)0.5, .5(.001)1 and 1(.01)5. The functions tabulated are p/p_0, ρ/ρ_0, T/T_0, a/a_0, q/a_*, A/A_*, $(p_0-p)/\frac{1}{2}\rho q^2$, $p_0/\frac{1}{2}\rho q^2$, and $(1+\gamma M^2)p/p_0$. (Here, by standard notation, p, ρ, T, a, q, A denote respectively pressure, density, temperature, sonic speed, flow speed, area; and the subscripts 0 and s indicate respectively the stagnation and sonic states.) In the supersonic range, the Mach angle μ is included. In Tables I.3 and I.4, the same ratios are referred to sonic state and tabulated in terms of q/a_* over the ranges 0(.01)2.44, 1(.001)1.5 and 1.5(.01)2.44. Table I.5 tabulates p/p_0, ρ/ρ_0, q/q_{max}, M, $\rho q/\rho_0 q_{max}$, $(p_0-p)/\frac{1}{2}\rho q^2$ as functions of $\tau=q^2/q^2_{max}$ for 0(.01)1 (max indicates the state attained by adiabatic expansion to zero pressure).

Part II concerns the relationships of the flow properties along the characteristics for both plane and axially symmetric cases. This consists of three tables: In Table II.1 μ, M, q/a_*, q/q_{max}, p/p_*, ρ/ρ_*, T/T_*, a/a_*, A/A_*, r/r_* (r being the radial distance from a point on the stream-line to the corner in Prandtl-Meyer expansion) are tabulated as functions of the angle ω of the flow referred to the sonic point over ranges 0°(0.5°)100° and 100°(1°)130°. In Table II.2, μ and M are given in ranges of ω for 0°(.001°).050°, 0°(.01°)1° and 0°(.1°)30°. In Table II.3, the functions given in II.1 and, in addition, the polar angle φ are tabulated with μ as independent variable over the range 90°(1°)0°.

Part III presents flow quantities behind both normal and oblique shocks as functions of both the Mach number M_1 and q_1/a_* in front of the shock. In Table III.1, p_2/p_1, ρ_2/ρ_1, T_2/T_1, p_{20}/p_{10}, p_2/p_{10}, p_1/p_{20} are tabulated as functions of M_1 over 1(.01)5 and in Table III.2, M_2 and $(q_1-q_2)/a_2$ are given over range of $M_1=1(.01)5$. (The subscript 2 refers to conditions behind the shock.) In Table III.3, the same ratios are calculated in terms of q_1/a_* over 1(.01)2.44. Table III.4 gives the maximum deflection angle δ_m of the flow after an oblique shock and the deflection δ_s at which the flow behind the shock is sonic as functions of M_1 over 1(.01)2 and 2(.1)5.

Part IV gives Mach number in terms of pressure ratios. In Table IV.1, M_1 is tabulated in terms of p_1/p_{10} (p_1, local pressure and p_{10}, stagnation pressure in free stream) over range 1(.001)0. In the case of supersonic flow, the pitot pressure p_{20} is different from p_{10}. The Mach number M_1 in terms of p_1/p_{20} over 0.528(.001).030 and p_{20}/p_{10} over 1(.001).030 are given in Tables IV.2 and IV.3. Tables V.1 and V.2 tabulate x^{n_1} and $(1-x^2)^{n_2}$ over $x=0(.005)1$ for $n_1=1/7, 2/7, 5/7, 1/5, 2/5, 7/5$ and $n_2=\pm7/2, \pm3, \pm5/2, \pm2, \pm3/2, \pm1, \pm1/2$. Part VI consists of 9 tables for the calculation of Reynolds numbers $R(M, T_0)$, pressure coefficient $C_p(M_1, T_0)$, the derivatives of the flow functions with respect to the ratio of the specific heats, and a table of standard atmosphere.

Each part is preceded by a brief explanation of the formulae employed and symbols involved. The selection of the material is excellent and is well suited to the workers in the field of aeronautical research. It will prove to be a valuable addition to the literature of aeronautics for a long time to come. *Y. H. Kuo* (Ithaca, N. Y.).

Shiffman, Max. On the existence of subsonic flows of a compressible fluid. J. Rational Mech. Anal. 1, 605–652 (1952).

The problem of the existence of steady irrotational flows of a compressible fluid satisfying prescribed boundary conditions is still unsolved in any generality. In the present paper the author makes a contribution to this problem. For two-dimensional flow about a smooth obstacle, he establishes that there is a constant M_0, depending on the obstacle, such that for every $M<M_0$ there exists a uniquely determined purely subsonic flow about the obstacle with Mach number M at infinity and with no circulation. The flow depends continuously on M, and the maximum local Mach number approaches 1 as $M\to M_0$. A similar result is shown for flows with circulation. This materially extends a result of Keldysh and Frankl [Izvestiya Akad. Nauk SSSR. Otd. Mat. Estest. Nauk (7) 1934, 561–601].

The discussion is based upon a regular variational problem for the stream function ψ of a subsonic flow. It is well known that ψ makes a certain integral stationary, but two important modifications of the integrand are necessary before a useful variational integral is obtained. The first provides for the convergence of the integral over the unbounded exterior of the obstacle by subtraction of an appropriate singularity (it is here that zero and non-zero circulation are distinguished). The second insures that the integrand, originally defined only for $|\text{grad}\,\psi|<b_0$ where b_0 depends on the Bernoulli constant, is regular for all values of ψ_x, ψ_y.

Direct methods of the calculus of variations are applied to the final variational integral. To obtain the desired existence theorem one must go a step beyond the usual proof of existence of a differentiable minimizing function ψ. In particular, it is necessary to discuss the behavior of the first derivatives of ψ on the boundary and at infinity, and to prove the continuous dependence of ψ on parameters. The work here, which is the main technical portion of the paper, is really an extension of the author's earlier work on the differentiability of solutions of variational problems [Ann. of Math. (2) 48, 274–284 (1947); these Rev. 9, 45]. The methods of the paper are immediately applicable to more general variational problems corresponding to elliptic partial differential equations other than that for subsonic flow. *J. B. Serrin* (Cambridge, Mass.).

Krzywoblocki, M. Z. E. Bergman's linear integral operator method in the theory of compressible fluid flow. Österreich. Ing.-Arch. 6, 330–360 (1952).

In the first part of his work on Bergman's linear integral operator method the author discusses subsonic flow. After deriving the Chaplygin's equation the author thoroughly discusses the fundamentals of Berman's method, the duality

between the flows of an incompressible and a compressible fluid as well as the problem of the convergence of the series expansion. A brief representation of the simplified pressure-density relation according to von Mises' proposition closes this part. *Author's summary.*

Truesdell, C. On curved shocks in steady plane flow of an ideal fluid. J. Aeronaut. Sci. 19, 826–828 (1952).

The work of T. Y. Thomas [J. Math. Physics 26, 62–68 (1947); these Rev. 8, 611] on the change in velocity gradient at a shock wave of given strength and curvature is extended to the case of a fluid with arbitrary thermodynamic properties. *M. J. Lighthill* (Manchester).

Phythian, J. E. The energy distribution behind a decaying two-dimensional shock. Quart. J. Mech. Appl. Math. 5, 318–323 (1952).

By approximating the non-uniform portions of unsteady one-dimensional or steady plane flows by means of simple waves, K. O. Friedrichs [Communications on Appl. Math. 1, 211–245 (1948); these Rev. 10, 638] has developed a theory of the formation and decay of shock waves in which terms of third order in shock strength have been neglected. Considering the work done by the piston and the energy in the simple wave, and making the hypothesis that errors due to certain third order changes in entropy and piston motion will be of the same order as those in Friedrichs's theory, M. J. Lighthill [Philos. Mag. (7) 41, 1101–1128 (1950); these Rev. 13, 400] has estimated for one-dimensional motion the errors in this theory due to third order errors in entropy and second order errors in the boundary conditions at the shock. The present paper summarizes a similar investigation for steady plane flow with results closely parallel to Lighthill's. To mention only two, there is a simple relation between the energy flux across any section x = constant of the residual wave (which follows the simple wave behind the shock) and the total entropy flux across the part of the shock upstream of this plane; and the third order pressure variations in the residual wave arise from pressure pulses associated with the reflection of the simple wave at the shock and subsequent reflection at the body. *J. H. Giese.*

McFadden, J. A. Initial behavior of a spherical blast. J. Appl. Phys. 23, 1269–1275 (1952).

At time $t=0$ a sphere of gas at uniformly high pressure is allowed to expand suddenly into a homogeneous atmosphere. The author evaluates the early stages of the motion before the head of the rarefaction wave has reduced its radius by a large fraction. He works from the zero-order solution given by the plane "shock tube" case, and calculates a first-order correction for the initial effects of sphericity. He gives also in principle the method for obtaining higher order corrections. A worked example, where a diatomic gas expands into a diatomic atmosphere, indicates (as one would expect) that after a short time the rarefaction wave is stronger and the shock wave weaker than in the plane case, so that the pressure distribution is everywhere lower and both the shock and the tail of the rarefaction move more slowly away from the initial interface. *M. J. Lighthill* (Manchester).

Niordson, Frithiof I. N. Transmission of shock waves in thinwalled cylindrical tubes. Trans. Roy. Inst. Tech. Stockholm 1952, no. 57, 24 pp. (1952).

A plane shock wave travels along a column of gas in an infinitely long thin-walled cylindrical tube. When the tube is perfectly rigid, simple relationships exist between the pressure, density, particle velocity, and shock velocity. However, in the present problem the tube is considered elastic. It deforms as the pressure varies and absorbs energy from the gas, which modifies the gas flow and shock. The author uses Fourier transform methods to calculate the elastic deformation and energy absorption of the cylinder when subjected to an internal pressure in the form of a step-function moving with the shock wave velocity. He then modifies the usual shock conditions to take into account this loss of energy in the gas, using an averaged value, for simplicity, and obtains the particle velocity behind the shock as a function of the shock velocity. A similar problem is solved when the tube changes abruptly, at a certain point, from a rigid section to an elastic section. *E. Pinney.*

Robinson, A. Non-uniform supersonic flow. Quart. Appl. Math. 10, 307–319 (1953).

A two-dimensional thin airfoil lies approximately in the plane of symmetry of a non-uniform (e.g., diverging or converging) supersonic stream whose components are $U(x, y)$, $V(x, y)$. The perturbations produced by the airfoil are calculated according to linearized theory. Both the main flow and the perturbations are supposed to be irrotational; it is also assumed that the velocity of sound can be taken constant. The resulting formulas for perturbation potential involve elliptic integrals, to which approximations for slightly convergent and slightly divergent streams are subsequently made. This process yields correction terms to be applied to familiar uniform-stream formulas. The lift and drag, in particular, are found to involve small corrections depending on the difference between Mach numbers at leading and trailing edges. *W. R. Sears* (Ithaca, N. Y.).

Duban, P. Solution graphique des écoulements plans supersoniques autour d'un dièdre. I. Texte et tableaux. O.N.E.R.A. Publ. no. 52, 135 pp. (1952).

Duban, P. Solution graphique des écoulements plans supersoniques autour d'un dièdre. II. Diagrammes. O.N.E.R.A. Publ. no. 52, 21 pp. (1952).

Stone, A. H. Corrections to the paper, "On supersonic flow past a slightly yawing cone II". J. Math. Physics 31, 300 (1953).

See same J. 30, 200–213 (1952); these Rev. 13, 702.

Jones, Robert T. Theoretical determination of the minimum drag of airfoils at supersonic speeds. J. Aeronaut. Sci. 19, 813–822 (1952).

In an earlier paper [same J. 18, 75–81 (1951); these Rev. 12, 554] the author derived general theorems concerning thin wings of minimum drag in frictionless flow. These are necessary and sufficient conditions involving the combined downwash $\bar{w}$, which is the sum of the downwash components for the lift (or thickness) distribution in question, in forward and reversed flow. There are no limitations on speed range or planform, except that no leading-edge singularities are considered.

Here an attack is made upon the problem of finding distributions that satisfy these conditions in supersonic flow. A general solution of the linearized potential equation is written in the form of a contour integral; viz.,

$$\varphi(x, y, z) = \oint F(\alpha x - \beta y - \gamma z, \lambda) d\lambda,$$

where α, β, and γ are functions of λ. In particular, the com-

bined field potential $\bar{\varphi}$ due to a given lift distribution is obtained in this form. It is shown that the combined downwash $\bar{w}$ will be constant over the planform, as required for minimum drag with given lift and planform, if the integrated loadings for certain oblique directions are elliptical. This is satisfied, for example, by a wing of elliptic planform with uniform lift distribution. The drag for such wings is therefore calculated, for various ratios of major to minor dimension and for various angles of yaw. The results are plotted.

Finally, the minimum drags of symmetrical (nonlifting) wings of elliptic planforms are calculated for the cases of given frontal area and given volume, respectively.

W. R. Sears (Ithaca, N. Y.).

Chapman, Dean R. Airfoil profiles for minimum pressure drag at supersonic velocities—general analysis with application to linearized supersonic flow. NACA Rep. no. 1063, ii+14 pp. (1952).

"Supersedes NACA Tech. Note no. **2264** (1951)"; these Rev. 13, 295.

Cohen, Doris. Formulas for the supersonic loading, lift, and drag of flat swept-back wings with leading edges behind the Mach lines. NACA Rep. no. 1050, iii+40 pp. (1951).

Berndt, Sune B. On the theory of slowly oscillating delta wings at supersonic speeds. Flygtekn. Försöksanstalt. Rep. no. 43, i+18 pp. (1 plate) (1952).

The author develops a method of attack on the boundary value problem associated with the harmonic motion of a subsonic leading edge delta wing that distorts in a generalized conical mode of shape $x^n f(y/x)$, where x and y are the chordwise and spanwise coordinates, and n is an integer. The known results for the rigid wing motions of plunging, pitching, and rolling are obtained, e.g. *J. W. Miles.*

Goldstein, Arthur W. Axisymmetric supersonic flow in rotating impellers. NACA Rep. no. 1083, ii+14 pp. (1952).

Supersedes NACA Tech. Note no. **2388** (1951); these Rev. 13, 86.

Cole, J. D., and Wu, T. Y. Heat conduction in a compressible fluid. J. Appl. Mech. 19, 209–213 (1952).

The diffusion of heat in the neighborhood of the characteristics for a non-viscous conducting compressible fluid is considered. To focus attention on the characteristics problem, the analysis is limited to small disturbances and the corresponding linearized forms of the general fluid behavior equations. By considering only irrotational flow the linearized equations of flow and heat conduction are essentially separable. The general case of an instantaneous point heat source is considered and provides a formal solution for certain large classes of initial conditions. In particular, the one-dimensional time-dependent case is solved explicitly in integral form with asymptotic expressions indicating the behavior at the point source origin and along the usual compressible flow characteristics. The total diffusion of heat is shown to be distributed between these two sets of characteristics with $1/\gamma$ of the total at the origin. *N. A. Hall.*

Yih, Chia-Shun. Laminar free convection due to a line source of heat. Trans. Amer. Geophys. Union 33, 669–672 (1952).

Closed solutions are given for the velocity and temperature distributions in a fluid in free convection above a line source of heat located in an infinite horizontal plane for Prandtl numbers 2/3 and 7/3. Variations of the kinematic viscosity and of the thermal diffusivity with temperature, as well as the heat generated by viscous dissipation, are neglected. The closed solution in terms of hyperbolic functions of the developed simultaneous ordinary differential equations is made possible by the special Prandtl numbers selected. *N. A. Hall* (Minneapolis, Minn.).

Høiland, Einar. On horizontal motion in a rotating fluid. Geofys. Publ. Norske Vid.-Akad. Oslo 17, no. 10, 26 pp. (1950).

An integral treatment of the velocity circulation for horizontal motions of the atmosphere leads the author to conclude that, neglecting baroclinicity, the vertical component of the absolute vorticity of an element of air is constant "following the motion". This conclusion also depends on the implicit assumption that the air is incompressible horizontally, so that the equation of continuity is satisfied by the introduction of a stream-function. After some general discussion of the phenomena in cyclones and anticyclones arising from the constancy of the absolute vorticity component, the author passes to motions in which the vertical component of absolute vorticity is proportional to the stream-function. The differential equation for this function is solved in a number of cases, allowance being made for the sphericity of the earth. The solutions are described in detail with the help of diagrams. A brief treatment of the impulse theorem is followed by solutions of a linearized equation for the wave-motions produced by perturbations of a constant zonal flow. Waves in a rectangular basin are also referred to. *G. C. McVittie.*

Sherman, Leon. The scalar-vorticity and horizontal-divergence equations. J. Meteorol. 9, 359–366 (1952).

Denoting by ζ the vertical component of the vorticity, and by D the horizontal divergence of the horizontal wind-components, formulae for $\partial\zeta/\partial t$ and $\partial D/\partial t$ are deduced from the three equations of motion. The terms in the formula for $\partial\zeta/\partial t$ are of five types, viz., pressure-force, viscous-force, advective, differential vertical-motion, and change-in-moment-of-inertia. The physical meanings of these five kinds of terms, and the reasons for the names given them, are expounded. The formula for $\partial D/\partial t$ contains six kinds of terms, viz., pressure-force, viscous-force, advective, differential vertical-advection, divergence, and virtual-force; these are elucidated in the same way as was done for the vorticity formula. A section is devoted to estimation of the orders of magnitude of the different kinds of terms.

G. C. McVittie (Urbana, Ill.).

Eliassen, Arnt. The quasi-static equations of motion with pressure as independent variable. Geofys. Publ. Norske Vid.-Akad. Oslo 17, no. 3, 44 pp. (1949).

In meteorological work, especially under the added assumption that the vertical pressure distribution is hydrostatic, it is often convenient to use pressure as a vertical coordinate. The author systematically transforms all the usual meteorological equations to such a system, and discusses their use and approximate solution. *C. Truesdell.*

Doak, P. E. The reflexion of a spherical acoustic pulse by an absorbent infinite plane and related problems. Proc. Roy. Soc. London. Ser. A. 215, 233–254 (1952).

A formal integral solution is given for the problem of the reflection of a spherical acoustic pulse by an infinite plane

interface having an impedance of arbitrary dependence on frequency and angle of incidence. In many cases of practical interest the impedance may be assumed to be independent of angle of incidence, and under this assumption the integral solution is relatively easy to evaluate. A simple exact expression for the reflected pulse, in closed form, is obtained when the wall impedance is purely resistive. To illustrate the relatively wide range of validity of the assumption of an impedance independent of angle of incidence, when applied to real materials, this exact result is compared with an approximate solution for the case where the interface separates two homogeneous isotropic lossless materials. The formal integral solution is evaluated approximately for wall impedances of several special types. The solutions are compared with corresponding solutions for plane incident waves, and the behaviour of the scattered wave, distinguishing between the spherical and the plane wave, is discussed. Possible applications of the results for acoustic waves to problems in the reflection of blast waves and of transient radiation by an electric dipole are indicated briefly. (From the author's summary.) *A. E. Heins* (Pittsburgh, Pa.).

Nomura, Yûkichi, and Kawai, Norio. On the acoustic field by a vibrating source arbitrarily distributed on a plane circular plate. Sci. Rep. Tôhoku Univ. Ser. I. 33, 197–207 (1949).

The author starts from the "wave equation" for the velocity potential with cylindrical coordinates ρ, φ, z and uses a solution which is a double infinite series of φ. The Fourier coefficients include infinite integral expressions involving Bessel functions of the first kind and exponential functions in the integrands. The series coefficients are determined by the velocity distribution on the circular plate. The above-said infinite integral expressions are evaluated using the addition theorem for Bessel functions. The acoustic field at a large distance from the circular plate is then calculated using Sonine-Gegenbauer formulas and asymptotic expressions for the Hankel functions. Finally, the pressure on the circular plate is calculated by the evaluation of the velocity potential on its surface. A formula for the radiation impedance is derived. Numerical values are given in several tables for the coefficients up to $k2\pi a/\lambda=4$, where a is the radius of the plate and λ the wavelength of sound in air.

M. J. O. Strutt (Zurich).

Nomura, Yûkichi, and Kawai, Norio. On the radiation of sound wave from a vibrating plane circular plate with a fixed circular baffle. Sci. Rep. Tôhoku Univ. Ser. I. 33, 208–215 (1949).

This paper pertains to the evaluation of the acoustic field caused by a circular plate vibrating axially in a fixed circular baffle. The velocity potential is expressed as an infinite series, the terms of which are infinite integrals involving products of Bessel functions of the first kind in their integrands. The coefficients are determined from the given distribution of velocity over the surfaces of the plate and the baffle. Expressions are obtained for the field at a large distance from the plate and this is evaluated numerically in two sets of curves and in a polar diagram. From the pressure on the plate's surface the radiation impedance is calculated and shown numerically in two sets of curves. The numerical values used in the diagrams are given in a table.

M. J. O. Strutt (Zurich).

Kawai, Norio. On the theory of Rayleigh disc. Sci. Rep. Tôhoku Univ. Ser. I. 35, 210–231 (1952).

In solving the problem of the diffraction of a plane acoustic wave by a circular disc, the author uses an expression for the velocity potential consisting of a double Fourier series, the terms of which contain infinite integrals involving the products of two Bessel functions of the first kind in their integrands. The circular disc is assumed to vibrate at the same frequency as the incident wave, the vibration being translational and rotational. From the boundary conditions on the disc's surface the coefficients of the above series are obtained. From this the total pressure on the disc is calculated. The resulting torque on the disc is then evaluated. Numerical values for these two expressions are then given up to $k=2\pi a/\lambda=4$, where a is the disc's radius and λ the wavelength in the surrounding medium at different angles of incidence. Two appendices and numerous tables and sets of curves complete the paper. Viscosity and turbulence are not accounted for. *M. J. O. Strutt* (Zurich).

Elasticity, Plasticity

***Westergaard, H. M. Theory of elasticity and plasticity.** Harvard University Press, Cambridge, Mass.; John Wiley & Sons, Inc., New York, 1952. xiii+176 pp. $5.00.

As is stated in the introduction (by G. M. Fair), the author had planned to write a more extensive textbook on elasticity and plasticity. The present volume contains the part that was completed at the time of the author's death in 1950.

In the first chapter some fundamental concepts are presented, and the relation of the theory of elasticity and plasticity to neighboring fields such as structural mechanics and soil mechanics is discussed. Chapter II contains a brief history of the field, which allows the author to introduce further fundamental concepts. Chapters III and IV are concerned with the mathematical specifications of stress and strain, Hooke's law, and a simple stress-strain law for a plastic material. In addition to the customary graphical representation of a state of stress by Mohr's circles, the dyadic circle of Land is used. Chapter V is devoted to the strain potential and its applications to problems concerning tubes and hollow spheres under internal pressure, rotating disks, and thermal stresses. Chapter VI introduces the Galerkin vector as a natural generalization of the strain potential. The author's twinned gradient is defined and applied to the investigation of the effect of a change in Poisson's ratio. A number of problems on the effect of a single force are treated.

The exposition is clear and careful, and the author's unconventional approach to many problems of elasticity and plasticity is stimulating. The reader will regret that the author did not live to complete a more comprehensive text in this vein. *W. Prager* (Providence, R. I.).

Rivlin, R. S. The solution of problems in second order elasticity theory. J. Rational Mech. Anal. 2, 53–81 (1953).

As is now known from examples, the general theory of elasticity predicts that new types of surface loads, not present at all in the classical theory, are required to effect a given deformation [see, e.g., §§10 and 42 of the reviewer's

paper, same J. **1**, 125–171, 173–300 (1952); these Rev. **13**, 794]. Conversely, it may be expected that given loads will produce new types of deformation. For example, a cylinder when subjected to a pure twisting couple will not only twist but also extend and change in volume. These new effects first make their appearance when terms of second order in the displacement gradients are considered. Some special approximate solutions of this type were obtained by Murnaghan [Finite deformation of an elastic solid, Wiley, New York, 1951, see Chs. 6 and 7; these Rev. **13**, 600]. Examples given by earlier writers are restricted to some special type of stress-strain relation. The author considers the fully general boundary-value problem of prescribed loads in the second order theory, in which there are five elastic constants, and in which all terms of third or higher order in the displacement gradients are systematically neglected. He proves that a solution (of course not unique) can always be obtained by the following process. (i) The displacements corresponding to the applied force system are first calculated on the basis of the classical elasticity theory. (ii) The additional forces which must be applied in order to maintain this deformation according to the finite deformation theory are calculated. These are, of course, of second order of smallness in the space derivatives of the displacement components. (iii) The displacements which these additional forces produce in an initially undeformed body according to classical elasticity theory are calculated. (iv) Addition of the displacements calculated in steps (i) and (iii) gives the displacements which are produced in the body by the initially applied forces." Steps (i) and (iii) each consist in solving a boundary value problem in linear elasticity theory. Step (ii) is an inverse problem. Hence the author's procedure is entirely effective, and solves completely the problem of exhibiting a solution. The position of this solution relative to the other possible solutions of the same boundary value problem remains to be established.

The author applies his method to the simultaneous torsion and extension of a hollow circular cylindrical tube and the simultaneous extension and torsion of a cylindrical rod of any cross-section (the solution being exhibited in terms of the classical torsion function, as suggested by the work of Green and Shield [Philos. Trans. Roy. Soc. London. Ser. A. **244**, 47–86 (1951); these Rev. **13**, 509], and Betti's method being used to determine the mean elongation). The author's procedure is not directly applicable to incompressible bodies, but at the end of the paper he constructs a limit process for specializing the results to the incompressible case.

C. Truesdell (Bloomington, Ind.).

Richter, Hans. Zur Elastizitätstheorie endlicher Verformungen. Math. Nachr. **8**, 65–73 (1952).

". . . wird das Hauptgewicht darauf gelegt, die verschiedenen bekannten Ergebnisse von einem einheitlichen geometrischen Standpunkte aus zu verknüpfen." (From the author's introduction.) *C. Truesdell.*

Hiedemann, E., und Spence, R. D. Zu einer einheitlichen Theorie der Relaxationserscheinungen. Z. Physik **133**, 109–123 (1952).

The authors attempt to unify and generalize the mathematical side of existing relaxation theories. Any use of the specific laws of mechanics is avoided. The first part, by the first author, discusses the philosophy of the subject. The second analytical part is by the second author. He considers the equation $E=KX$, where "X and E stand for any complex stress and the accompanying strain, both depending on time as $e^{i\omega t}$," and where K is a "complex modulus of elasticity," a function of ω. He replaces $E=KX$ by a corresponding complex Fourier integral representation of $E(t)$ in terms of X. He next assumes that if $X=0$ for $t<t_0$, then also $E=0$ for $t<t_0$, and conversely; and that if X is real, so also is E, and conversely. He then uses function-theoretic methods to discuss the consequences of these and other restrictions on the behavior of $E(t)$. Thus he obtains various types of "relaxation spectra." *C. Truesdell.*

Pieruschka, E. Ein Stoffgesetzansatz für elastische, anisotrope Medien. Ing.-Arch. **20**, 229–233 (1952).

The author proposes a certain restricted type of anisotropy for the linear theory of elasticity. Instead of the usual 36 (or 21) constants, the author's theory contains 7. He does not discuss the existence of a strain energy in this case. He obtains the equations for propagation of infinitesimal waves in such a material. *C. Truesdell.*

Grioli, Giuseppe. Relazioni quantitative per lo stato tensionale di un qualunque sistema continuo e per la deformazione di un corpo elastico in equilibrio. Ann. Mat. Pura Appl. (4) **33**, 239–246 (1952) = Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo no. **335** (1952).

Let X_{ij} be the components in a rectangular system x_i of any tensor defined over a volume V whose boundary is S. The author defines the vector fields

$$b^{(r)}_{\eta\tau\lambda}\equiv-\frac{1}{V}\left\{\int_V x_1{}^\eta x_2{}^\tau x_3{}^\lambda X_{r i, i}dV+\int_S x_1{}^\eta x_2{}^\tau x_3{}^\lambda X_{ri}dS_i\right\}.$$

When X_{ij} is the stress field for a medium of known density in equilibrium, by Cauchy's equations the right-hand side can be calculated in principle from the applied loads, so that the moments $b^{\cdot}_{\cdots}$ become known a priori, even though the stress field X_{ij} is not known throughout V and in fact is not even determinate unless constitutive equations are added. The set of fields $b^{\cdot}_{\cdots}$ for which $\eta+\tau+\lambda=n$ the author calls n-hyperstatic co-ordinates. The cases $n=1$, 2 have appeared in the earlier Italian literature. After expressing the b's in terms of combinations of the moments of X_{ij} over V, the author selects particularly simple ones: $\beta^{(r)}_{sn}$ is obtained by putting n for the exponent of x_s and 0 for the other two; $\gamma^{(r)}_{sn}$ by putting $n-1$ for the exponent of x_r, 1 for that of x_s, and 0 for the third. These particular b's are expressed in terms of moments of a single one of the X_{ij}. Hence follow the general bonds

$$|X_{rs}|_{\max}\geqq\frac{V}{n\int_V|x_s|^{n-1}dV}|\beta^{(r)}_{sn}|,\quad n\geqq1,$$

$$|X_{rr}|_{\max}\geqq\frac{V}{(n-1)\int_V|x_r|^{n-2}|x_s|dV}\left|\gamma^{(r)}_{sn}-\frac{1}{n}\beta^{(s)}_{rn}\right|,\quad n\geqq2.$$

The remainder of the paper concerns applications of these bounds to the linear theory of elasticity. Hence the author is able to generalize an inequality of Signorini [Ann. Scuola Norm. Super. Pisa (2) **2**, 231–251 (1933)] to obtain lower bounds for the maximum strains and displacements, and to derive some extremal theorems. *C. Truesdell.*

Grioli, Giuseppe. Proprietà di media ed integrazione del problema dell'elastostatica isoterma. Ann. Mat. Pura Appl. (4) 33, 263–271 (1952) = Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo no. **337** (1952).

The author remarks that while it is often more valuable to determine stress than displacement, the usual approximative procedures in linear elasticity theory obtain the latter first, thereafter yielding the former by slowly convergent processes arising from differentiation. In the paper reviewed above he has obtained expressions for certain moments of the stress in terms of quantities known a priori in problems where the applied loads are prescribed. From these he obtains means of the products of the stress components by certain polynomials forming a complete orthogonal set over the body. Hence, by applying a procedure of Amerio [Amer. J. Math. **69**, 447–489 (1947); these Rev. **9**, 37], he is able to set up and prove the convergence of an iterative procedure in terms of the stress alone. *C. Truesdell.*

Grioli, G. Integrazione del problema della statica delle piastre omogenee di spessore qualunque. Ann. Scuola Norm. Super. Pisa (3) **6**, 31–49 (1952) = Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo no. **339** (1952).

The method of the previous paper is applied to the theory of flat plates. When the plate is thin, it is particularly appropriate, since only mean values are desired. To apply it here he assumes the stress linearly distributed across the plate. While the author's method is worked out for anisotropic materials, in the isotropic case he uses it to derive the usual approximate formulae. *C. Truesdell.*

Grioli, Giuseppe. Validità del teorema di Menabrea e integrazione del problema dell'elastostatica in casi non isotermi. Rend. Sem. Mat. Univ. Padova **21**, 202–208 (1952).

The author extends the theorem of Menabrea to certain types of linear thermoelastic deformations. Then he uses an extension of the method of the paper cited in the third review above to obtain a lower bound for the temperature change in such a deformation. *C. Truesdell.*

Arf, C. On the determination of multiply connected domains of an elastic plane body, bounded by free boundaries with constant tangential stresses. Amer. J. Math. **74**, 797–820 (1952).

The contents of this paper, which is rather involved, may best be summarised by the author's formulation of the problem as given in paragraph 1 of the paper.

"B. Consider the closed curves L on the z-plane which are boundaries of multiply connected domains C of multiplicity m such that there exists a state of stress of C which satisfies the following conditions. B_1. The stress tensor $(\sigma_x, \sigma_y, \tau)$ satisfies the condition $\sigma_x+\sigma_y=4\alpha$ with a given constant α throughout C and on L. B_2. The stress tensor $(\sigma_x, \sigma_y, \tau)$ has determined finite or infinite values everywhere in C and on the boundary L, with the possible exception at $z=x+iy=\infty$. More precisely, the analytic function $\Gamma(z)=(\sigma_x-\sigma_y-2i\tau)/4\alpha$ has no singularities other than poles in C and on the boundary L except at the point $z=\infty$. B_3. The boundary L is a free boundary for the state of stress $(\sigma_x, \sigma_y, \tau)$. B_4. $\Gamma(z)$ has only a finite number of poles in C and on L. Besides these we shall require that the following geometrical conditions are satisfied. B_5. The domain C does not contain the point $z=\infty$. But the point $z=\infty$ might be a point of the boundary L. For a variable point z of the domain which tends towards ∞ continuously, $\arg z$ has limits which are the numbers $\phi_1, \phi_2, \cdots, \phi_n$ $(0\leqq\phi_1<\phi_2<\cdots<\phi_n<2\pi)$. B_6. To every ϕ_j is associated a set of separated intervals

$$(a_1^{(j)}, b_1^{(j)}),\ (a_2^{(j)}, b_2^{(j)}),\ \cdots,\ (a_{\nu_j}^{(j)}, b_{\nu_j}^{(j)}),$$

which are described by the expression $\lim_{z\to\infty} \Im z \exp(-i\phi_j)$, where $z \in C$ is supposed to go to infinity continuously and $\lim_{z\to\infty} \arg z=\phi_j$. B_7. L has only a finite number of points of inflexion and a finite number of cuspidal points. . . . The purpose of this paper is to determine the domains C and the stress tensors σ_x, σ_y, τ which satisfy the conditions above."

As a special case the author determines C when $(\sigma_x-\sigma_y-2i\tau)/4\alpha$ is finite everywhere in C. In addition, he determines (a) the resultant $X+iY$ of the forces which are exerted on the boundary of a small body containing the point z_j where $(\sigma_x-\sigma_y-2i\tau)/4\alpha$ becomes infinite, (b) the resultant momentum of these forces with respect to the point z_j, and (c) the displacement corresponding to the given state of stress. *R. M. Morris* (Cardiff).

Yu, Yi-Yuan. Gravitational stresses on deep tunnels. J. Appl. Mech. **19**, 537–542 (1952).

Using Mushelišvili's complex variable method for solving two-dimensional elasticity problems, the gravitational stresses around a horizontal tunnel opening are determined. The material surrounding the tunnel is assumed to be elastic, isotropic, and homogeneous, and it is shown, in the first place, that the body force throughout the material may be considered as equivalent to a surface force. The tunnel is located at a large but finite depth underneath the horizontal ground surface and has the shape of a general ovaloid, including the rounded cornered square, the ellipse, and the circle as special cases. Two problems are solved. In the first, an unlined tunnel is considered which has a boundary free from external stresses, whilst the second is a tunnel with a rigid lining, and a perfect bond is assumed between the lining and the surrounding material, so that the displacements at the boundary are zero. In the first case the hoop stress at the boundary is determined for the general ovaloid, and in the second case the three stress components at the boundary are determined for the circular tunnel. Numerical results are given. *R. M. Morris* (Cardiff).

Kalandiya, A. I. The general mixed problem of bending of an elastic plate. Akad. Nauk SSSR. Prikl. Mat. Meh. **16**, 513–532 (1952). (Russian)

In the present paper the author applies the method of singular integral equations to the determination of the deflection of an elastic plate, when its boundary is partly clamped, partly simply supported, and partly free. (In a previous paper [Kalandiya, same journal **16**, 271–282 (1952); these Rev. **14**, 334] the problem of the determination of deflection of an elastic plate, partly clamped and partly simply supported along the boundary, had been reduced to equivalent Fredholm integral equations.) The boundary value problem considered here is the following:

$$\Delta\Delta u=q/D_0,\ \text{in } T;\quad u=du/d\nu=0,\ \text{on } L^{(1)};$$

$$u=0,$$

$$G(u)\equiv\sigma\Delta u + (1-\sigma)[u_{xx}\cos^2\theta+2u_{xy}\cos\theta\sin\theta+u_{yy}\sin^2\theta]=0, \quad \text{on } L^{(2)};$$

$$G(u)=0,$$

$$\frac{d\Delta u}{d\nu}+(1-\sigma)\frac{d}{ds}[u_{xy}\cos 2\theta+(u_{yy}-u_{xx})\cos\theta\sin\theta]=0, \quad \text{on } L^{(3)};$$

here T is a bounded open plane set, $L=L^{(1)}+L^{(2)}+L^{(3)}$ is its (sufficiently smooth) boundary, q is the given external load, Δ is the Laplacian, D_0 an elastic constant, σ is Poisson's ratio, and θ is the angle between the outer normal to L and the x-axis. The argument rests upon results of Mushelišvili [Singular integral equations, Moscow-Leningrad, 1946; these Rev. **8**, 586], N. P. Vekua [Systems of singular integral equations, Moscow-Leningrad, 1950; these Rev. **13**, 247], and D. I. Šerman [Doklady Akad. Nauk SSSR (N.S.) **27**, 911–913 (1940); these Rev. **2**, 270; Akad. Nauk SSSR. Trudy Seĭsmol. Inst. **88**, 1–32 (1938)]. *J. B. Dias.*

Kudryavcev, N. V. Bending of a circular plate with a central opening under the action of a concentrated force. Doklady Akad. Nauk SSSR (N.S.) **50**, 111–115 (1945). (Russian)

Huang, M. K., and Conway, H. D. Bending of a uniformly loaded rectangular plate with two adjacent edges clamped and the others either simply supported or free. J. Appl. Mech. **19**, 451–460 (1952).

Employing superposition methods the authors solve the bending of uniformly loaded rectangular plates when (A) two adjacent edges are clamped and the others are simply supported and (B) two adjacent edges are clamped and the others are free. The essence of the method is to replace the given boundary problem by another one involving a boundary, part of which coincides with the original boundary, and introducing sufficient parameters in the new problem solution in order to satisfy the required boundary conditions for the original problem. In (A) the new region is a new rectangle twice as large in each dimension as the original and with simply supported edges and uniformly loaded. Two line loads are imposed on the axes of symmetry to bring them to zero deflection resulting in having each quadrant of the new plate equivalent to the original problem. This method was used by Sapondžyan [Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 501–512 (1949); these Rev. **11**, 287] for problem (A) and a variety of other cases. In (B) the authors superpose five new problems, but the reviewer questions whether the corner condition $w_{xy}=0$ is satisfied at the free corner. *D. L. Holl* (Ames, Iowa).

Uflyand, Ya. S. Application of the Mellin transform to the problem of the bending of a thin elastic plate in the shape of a wedge. Doklady Akad. Nauk SSSR (N.S.) **84**, 463–465 (1952). (Russian)

The Mellin transform

$$\bar{w}(\theta, \nu)=\int_0^\infty w(r, \theta)r^{\nu-2}dr, \quad 1-\beta<\operatorname{Re}\nu<1+\gamma,$$

is used to determine the deflection $w(r, \theta)$ of a thin elastic plate in the shape of a wedge: $0\leqq r<\infty$, $0\leqq\theta\leqq\alpha$, which satisfies the equation

$$\nabla^4 w=q(r, \theta)/D \quad (D=Eh^3/12(1-\sigma^2))$$

(it is assumed that as $r\to\infty$, $w=O(r^{-\gamma})$, where $\gamma>0$, and as $r\to 0$, $w=O(r^\beta)$, where $\beta>1$). The boundary conditions are of two kinds: (a) simple support along both edges,

$$w|_{\theta=0}=w|_{\theta=\alpha}=d^2w/d\theta^2|_{\theta=0}=d^2w/d\theta^2|_{\theta=\alpha}=0;$$

(b) simple support along one edge and clamped along the other edge,

$$w|_{\theta=0}=dw/d\theta|_{\theta=0}=w|_{\theta=\alpha}=d^2w/d\theta^2|_{\theta=\alpha}=0.$$

J. B. Dias (College Park, Md.).

Galimov, K. Z. Equilibrium equations of the theory of elasticity for finite displacements and their application to the theory of shells. Izvestiya Kazan. Filial. Akad. Nauk SSSR. Ser. Fiz.-Mat. Tehn. Nauk **1**, 25–46 (1948). (Russian)

The author considers the geometry of finite strain and proceeds to express the equations of equilibrium in the form

$$A^{\alpha\beta}{}_{,\alpha}+\rho_0F^\beta=0,$$

where $A^{\alpha\beta}$ is the contravariant component in the direction of the undeformed axis x^β of the stress vector acting on the deformed surface referred to unit area of the undeformed surface, which, as a result of deformation, goes into the deformed surface $x^\alpha=$ constant. This tensor is not symmetric but presents the advantage that the covariant derivation, indicated by the comma, depends only on the metric tensor of the undeformed state. The introduction of this tensor, in the theory of shells, leads to an additional term in the moment equation of equilibrium. He concludes that the momentless state in shells with finite displacements is practicable only when certain restrictive conditions are put on the displacements. The shell of variable thickness is next considered and equations are derived which have already been given by Chien [Quart. Appl. Math. **1**, 297–327 (1944); these Rev. **5**, 195]. The method of derivation is, however, different. *L. M. Milne-Thomson* (Greenwich).

Galimov, K. Z. The general theory of elastic shells with finite displacements. Izvestiya Kazan. Filial. Akad. Nauk SSSR. Ser. Fiz.-Mat. Tehn. Nauk **2**, 3–38 (1950). (Russian)

This is a continuation of the work of the paper reviewed above. The first section gives the theory of the deformation of surfaces in general coordinates, and conditions of compatibility. The second gives the equilibrium equations of shells of undeformed thickness $2h$ referred to the metric of the undeformed shell. For this purpose the author introduces a moment tensor analogous to the stress tensor $A^{\alpha\beta}$ of the first paper, and the relation between the deformation and the external force system is examined. [The author claims to make no arbitrary hypotheses but it seems to the reviewer that he tacitly assumes that the undeformed middle surface and its normals go over to the deformed middle surface and its normals.] The third section introduces boundary conditions in forms adapted to the deformed and undeformed states. In the fourth section the equations are formulated with the approximation of Kirchhoff's hypothesis. The fifth section gives the full set of equations of equilibrium and compatibility for thin shells. [This discussion appears to be not entirely satisfactory, for the author gives no explicit definition of "thin".] *L. M. Milne-Thomson.*

Clark, R. A., and Reissner, E. A problem of finite bending of toroidal shells. Quart. Appl. Math. **10**, 321–334 (1953).

The authors analyze the toroidal expansion joint for two straight sections of a cylindrical shell, the loading being axisymmetrical. The linearized theory of such joints has already been given [R. A. Clark, J. Math. Physics **29**, 146–178 (1950); these Rev. **12**, 557] and the object of the present work is to establish the range of validity of this theory and to extend it to include slight non-linearity. The differential equations used are those developed by the second author [Proc. Symposia Appl. Math., vol. 3, McGraw-Hill, New York, 1950, pp. 27–52; these Rev. **12**, 557]. Approximate solutions of these are obtained for $\mu\gg 1$

{$\mu=[12(1-\nu^2)]^{1/2}(b^2/ah)$, where b is the radius of the cross-section of the middle surface of the shell, a is the radius of the center line of the torus, and h is the wall thickness} and data are given for the determination of the maximum stresses and relative axial displacement of the joint. It is noted that displacements much larger than the wall thickness are adequately covered by the linearized theory.

H. D. Conway (Ithaca, N. Y.).

Federhofer, Karl. Zur Berechnung zylindrischer Behälter mit veränderlicher Wandstärke. Anz. Öster. Akad. Wiss. Math.-Nat. Kl. **1950**, 275–287 (1950).

The equilibrium and elastic deformation equations for a cylindrical vessel with axis vertical, varying wall thickness, and pressure due to a contained heavy liquid are developed. A fourth-order or pair of second-order differential equations result. For the case when the wall thickness is given as a second-degree polynomial in the height, a closed form solution is obtained with four arbitrary constants, which can be chosen to satisfy boundary conditions at the top and bottom.

E. H. Lee (Providence, R. I.).

Mikeladze, Š. E. General equation of the elastic line of a beam. Doklady Akad. Nauk SSSR (N.S.) **50**, 117–119 (1945). (Russian)

Nuttall, Henry. Torsion of uniform rods with particular reference to rods of triangular cross section. J. Appl. Mech. **19**, 554–557 (1952).

The author employs a set of eigenfunctions $\phi_n(x, y)$ satisfying $\nabla^2\phi_n=-\lambda_n^2\phi_n$ to represent the Prandtl torsion function $\psi=\sum A_n\phi_n$. Since ψ is to vanish on the boundary of the prism, each ϕ_n also is required to vanish. The coefficients A_n are determined either by minimizing the total potential integral or by minimizing the integral of mean square error of the required differential equation which ψ is to satisfy. Exact results are obtained for the equilateral triangle and right-angled isosceles triangle by the use of trigonometric functions as eigenfunctions. For the general isosceles triangle the approximate results for maximum shear and moment are compared with those obtained by relaxation methods.

D. L. Holl (Ames, Iowa).

Das Gupta, Sushil Chandra. Note on Love waves in a homogeneous crust laid upon heterogeneous medium. J. Appl. Phys. **23**, 1276–1277 (1952).

In the first case discussed in this paper a homogeneous layer is supposed to be underlaid by a half-space where the density and rigidity vary according to an exponential law. The corresponding frequency equation differs in one simple factor from that for regular Love waves. In the second case the heterogeneous substratum is of the type considered by Meissner (1921). The solution of the equation for transverse waves in such a medium is now obtained in terms of Whittaker functions and these functions appear in the frequency equation. An approximate form of this equation is derived for very slow variations in density and rigidity as functions of the depth.

W. S. Jardetzky.

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

Thomescheit, A. Differenzenformeln zur Berechnung der optischen Bildfehler meridionaler Strahlenbüschel mit besonderer Berücksichtigung telezentrischen Strahlenganges im Bildraum. Optik **9**, 360–378 (1952).

The Kerber difference formulas for the intersection distances in the case of an axial object point are here extended to the case of aberration-afflicted objects, consideration being given to the case of telecentric ray-passage in the image space and to the case of plane surfaces. A procedure is described for applying the results to systems where a refracting surface is bounded by more than two media. Precise summation formulas are derived for the intersection angles of the rays with the axis in cases where the ray-passage in the image is telecentric and the intersection angles in the object space are afflicted with aberrations. The relations obtained were used to derive difference formulas for lateral magnification, focal length, and other characteristic numbers and are compared with the results of A. Kerber and with the difference formulas of F. Staeble for the correction of the isoplanary condition. In a similar fashion, the known difference formulas for lateral and longitudinal paraxial color errors were generalized to the case of telecentric ray passage in the image space and an aberration-afflicted object. The generalized monochromatic difference formulas were used to derive an efficient method of tracing meridian rays from an off-axis point, comparison being made here with the results of F. Temmermann and of M. Herzberger.

E. W. Marchand (Rochester, N. Y.).

Mikaèlyan, A. L. Application of the method of coordinate systems for construction of nonhomogeneous media with given trajectories of rays. Doklady Akad. Nauk SSSR (N.S.) **86**, 1101–1103 (1952). (Russian)

Il piano viene riferito alle coordinate curvilinee ortogonali u_1, u_2 i raggi assegnati essendo rappresentati dalle linee $u_2=$costante. Viene applicato il principio di Fermat e la relativa equazione di Eulero. Se l'elemento lineare nelle coordinate curvilinee prescelte è espresso da $ds^2=h_1^2du_1^2+h_2^2du_2^2$, la velocità di fase $v(u_1, u_2)$ nel mezzo non omogeneo risulta data da $v(u_1, u_2)=h_1(u_1, u_2)S(u_1)$, essendo S una funzione arbitraria. Si può quindi calcolare la distribuzione dell'indice di rifrazione.

G. Toraldo di Francia (Firenze).

Regenstreif, Edouard. Théorie du régime transgaussien de la lentille électrostatique elliptique. Ann. Radioélec. **6**, 299–317 (1951).

The purpose of this paper is to carry out the integration of the differential equations determining the paths followed by the transgaussian rays in the elliptical electrostatic lens and to establish explicit formulae for the transgaussian optical elements of this lens. *From the author's summary.*

Boersch, H. Über die Gültigkeit des Babinetschen Theorems. Z. Physik **131**, 78–81 (1951).

The author is concerned with the proof of the elementary form of Babinet's theorem, that for Fraunhofer diffraction. The theorem states that given complementary screens with identical incident illumination, the amplitude of the complex values u_1 and u_2 of the excitation behind the screens fulfill

the equation $|u_2|=|u_3|$. The proof of this proceeds from the more generally valid equation $u_1=u_2+u_3$, where u_1 is the complex excitation with the screens removed. The necessary argument for the Fraunhofer case is that $u_1=0$. The author shows, however, by studies of both rectangular and circular apertures, that, in general, $u_1 \nrightarrow 0$ as the apertures become large. He gives restrictive conditions under which u_1 does vanish and the usual proof is presumably valid. *W. K. Saunders* (Washington, D. C.).

Jones, D. S. Diffraction by an edge and by a corner. Quart. J. Mech. Appl. Math. **5**, 363–378 (1952).

Conditions are given which are sufficient to ensure that the current density normal to an edge is zero at that edge. An extra condition is given which makes the components of the field which are parallel to the edge finite. The solution is then shown to be finite. Simpler conditions are given for two-dimensional fields. The agreement of various known solutions with these assumptions are discussed. It is finally shown that certain simple current densities lead to a unique solution for the diffraction by corners of flat surfaces.
A. E. Heins (Pittsburgh, Pa.).

Franz, W., und Deppermann, K. Theorie der Beugung am Zylinder unter Berücksichtigung der Kriechwelle. Ann. Physik (6) **10**, 361–373 (1952).

The authors deal with the diffraction of scalar and electromagnetic waves by infinite cylindrical bodies. Emphasis is upon cylinders of a size which is too small for a successful application of the theory of geometrical optics and too large to permit calculation of the Debye series. The attack is by the use of integral equations formulated by Maue [Z. Physik **126**, 601–618 (1949), p. 606; these Rev. **11**, 293] and others. In using the integral equation for the lighted side it is assumed that at the surface of the cylinder the phase of the secondary excitation is identical to that of the primary excitation. On the shadow side the assumption is made that the secondary excitation is in the form of waves progressing around from each of the shadow boundaries (Kriechwelle). These are solutions of the homogeneous form of the integral equation and may be added to the solution for the lighted side as well. The two integral equation solutions are matched at the shadow edge. Formulas for both incident polarizations are given, and a comparison with experimental results is made. [Reviewer's note. Many of the details of the paper are difficult to follow due to a somewhat cryptic presentation coupled with several misprints. Formulas (1) and (12) are clearly the result of jumbling by the printer but (15) is a more serious obstacle as its derivation is less apparent and the integral as written does not exist.] *W. K. Saunders.*

Bekefi, G. Diffraction of electromagnetic waves by an aperture in an infinite screen. J. Appl. Phys. **23**, 1403 (1952).

Woonton, G. A. Diffraction field of a circular aperture. J. Appl. Phys. **23**, 1405–1406 (1952).

Neugebauer, Hans E. J. A new method of solving diffraction problems. J. Appl. Phys. **23**, 1406 (1952).

Mirimanov, R. G. A new method of solving problems of the reflection of electromagnetic waves from thin nonclosed surfaces of finite curvature. Defence Scientific Information Service, Defence Research Board, Ottawa, Canada, Rep. **T68R**, 7 pp. (undated).

Translated from Doklady Akad. Nauk SSSR (N.S.) **66**, 641–644 (1949); these Rev. **10**, 764.

Mirimanov, R. G. The diffraction of a spherical electromagnetic wave from a paraboloid of revolution of limited size, when the dipole exciting the field lies along the axis of symmetry of the paraboloid. Defence Scientific Information Service, Defence Research Board, Ottawa, Canada, Rep. **T67R**, 6 pp. (1 plate) (undated).

Translated from Doklady Akad. Nauk SSSR (N.S.) **67**, 835–838 (1949); these Rev. **11**, 142.

Twersky, V. On a multiple scattering theory of the finite grating and the Wood anomalies. J. Appl. Phys. **23**, 1099–1118 (1952).

The author gives a complete solution for the Hertz vector potential due to an incident plane wave scattered by a finite grating of parallel cylinders. This solution is then specialized for distances large compared with the grating width and to spacings larger than the wavelength. Neglecting end effects, an explicit solution is obtained. Further discussion is facilitated by deriving approximate solutions, restricting the problem to radii very small compared to the wavelength. For a plane wave polarized parallel to a cylinder, or for a cylinder of zero impedance in acoustics, using only the largest coefficient, numerical expressions are obtained and sketched in figures. Detailed discussions of these approximations follow, supported by a number of graphs. The next approximation is then added to the first one and the sum is discussed. Then the reflection by a grating of bosses on a plane is treated and also discussed in detail, supported by further graphs. Some derivations, left out in the text, are given in an appendix. *M. J. O. Strutt* (Zürich).

Ronchi, Laura, et Toraldo di Francia, Giuliano. Application du calcul des variations à la détermination des coefficients de réflexion. Rev. Optique **31**, 481–484 (1952).

The reflection coefficients in a stratified medium are computed with the aid of variational principles.
A. E. Heins (Pittsburgh, Pa.).

Wait, James R. Reflection of electromagnetic waves obliquely from an inhomogeneous medium. J. Appl. Phys. **23**, 1403–1404 (1952).

Müller, Claus. Zur Methode der Strahlungskapazität von H. Weyl. Math. Z. **56**, 80–83 (1952).

An alternate proof of the scalar radiation existence theorem as proved by H. Weyl [Math. Z. **55**, 187–198 (1952); these Rev. **14**, 225]. The author obviates the construction of manifolds of homogeneous solutions of the iterated operators by showing that the "capacity" matrix of the solutions of the homogeneous integral equation and its adjoint is itself not singular. The necessary tool is the use of additional results proved by Rellich [Jber. Deutsch Math. Verein. **53**, 57–65 (1943); these Rev. **8**, 204].
W. K. Saunders (Washington, D. C.).

Agostinelli, Cataldo. Sopra due casi notevoli di integrabilità delle equazioni della propagazione di onde elettromagnetiche in un tubo cilindrico circolare con dielettrico eterogeneo. Boll. Un. Mat. Ital. (3) **7**, 267–272 (1952).

The author considers the propagation of transverse electric (TE) and transverse magnetic (TM) waves within a perfectly conducting, circularly cylindrical waveguide filled with a heterogeneous dielectric. Under the assumptions that a) the dielectric constant varies parabolically with the distance r from the axis as $\epsilon=\epsilon_1+(\epsilon_2-\epsilon_1)r^2/a^2$, where a is the

radius of the cylinder, b) the fields depend on time as $\exp(i\omega t)$ and on the longitudinal coordinate z as $\exp(-i\beta z)$, he finds that the source-free Maxwell equations for waves propagating in the longitudinal direction with the critical phase velocity $V=c/(\mu\epsilon_1)^{1/2}$, where c and μ are the velocity of light and the permeability, lead to

$$(1)\qquad \frac{d^2\mathcal{E}_3}{d\epsilon^2}+\frac{1}{\epsilon}\frac{d\mathcal{E}_3}{d\epsilon}+\frac{\beta^2a^2}{4\epsilon_1(\epsilon_2-\epsilon_1)}\mathcal{E}_3=0$$

for TM waves, and

$$(2)\qquad \frac{d^2\mathcal{H}_3}{d\epsilon^2}+\frac{\beta^2a^2}{4\epsilon_1(\epsilon_2-\epsilon_1)}\mathcal{H}_3=0$$

for TE waves. $\mathcal{E}_3$ and $\mathcal{H}_3$ are the covariant, longitudinal components of the electric and magnetic vectors. The solutions of (1) are the Bessel and Neumann functions of zero order, and those of (2) are the sine and cosine. Infinite sets of propagation constants for the TM and TE waves are found by applying to the solutions of (1) and (2) the conditions, $\mathcal{E}_3=0$ and $d\mathcal{H}_3/dr=0$ at the boundary $r=a$. In the derivation of (1) and (2) the transverse coordinates are chosen to be isometric (or isothermic). *C. H. Papas.*

Ortusi, J., et Simon, J.-C. Les ondes principales dans les guides électromagnétiques. Ann. Radioélec. **5**, 12–20 (1950).

After describing the application of Maxwell's equations to guided waves, this paper is concerned with the general expression of the transverse electromagnetic wave as defined by a velocity of propagation equal to that of light in unbounded space. It is shown that the characteristic functions of such waves (power, voltage, current, characteristic impedance) are invariant in any conformal transformation of coordinates. The latter theorem is then used to evaluate the iterative impedances of coaxial and multi-wire lines from the characteristic impedance of two strips of unbounded plane, which corresponds to the simplest T.E.M. wave. *Author's summary.*

Hafner, E. Das vollständige System der elektromagnetischen Eigenschwingungen eines zweiachsig anisotropen Parallelepipeds. Acta Physica Austriaca **6**, 209–218 (1952).

Maxwell's field equations are solved for the interior of a parallelepiped cavity filled with an optically biaxial lossless dielectric. The coordinate axes are taken in the directions of the axes of the dielectric tensor ellipsoid and the boundary conditions require vanishing tangential electric field components on the walls of the cavity. It is shown that the complete solution is obtained by the superposition of two Hertz vector potentials each with two components; one Hertz vector gives the electric, the other the magnetic type modes. *E. Weber* (Brooklyn, N. Y.).

Finzi, Bruno. Discontinuità dei campi elettromagnetici nello spazio-tempo. Boll. Un. Mat. Ital. (3) **7**, 252–259 (1952).

Let $\tau(x^\alpha)=$constant be the characteristic hypersurfaces (defining the wave-fronts) of electromagnetic phenomena in a galilean space-time. For the discontinuity in the derivatives of the electromagnetic bivector $F_{\alpha\beta}$ across the hypersurfaces, the author writes

$$DF_{\alpha\beta/\gamma}=\Lambda_{\alpha\beta}\tau_{/\gamma},$$

where / denotes covariant differentiation and the bivector $\Lambda_{\alpha\beta}$ is a priori arbitrary. The field equations give

$$\Lambda_{\alpha\beta}\tau^{/\beta}=0,\quad {}^*\Lambda_{\alpha\beta}\tau^{/\beta}=0,$$

${}^*\Lambda_{\alpha\beta}$ being the dual bivector, and the fact that τ satisfies the partial differential equation $\tau^{/\beta}\tau_{/\beta}=0$ quickly follows. A similar formulation may be made for the field inside matter. Here there are the two bivectors $F_{\alpha\beta}$, $f_{\alpha\beta}$, the former representing electric force and magnetic induction and the latter magnetic force and electric displacement. If it is assumed that these are connected by a linear relation $f_{\alpha\beta}=C_{\alpha\beta\rho\sigma}F^{\rho\sigma}$, with $C_{\alpha\beta\rho\sigma}=-C_{\beta\alpha\rho\sigma}=C_{\rho\sigma\alpha\beta}$, the differential equation satisfied by τ is det $|C^{\alpha\beta\rho\sigma}\tau_{/\beta}\tau_{/\rho}|=0$. This takes a special form in the case of a homogeneous crystal. The methods are also applicable to fields more general than Maxwellian. *H. S. Ruse* (Leeds).

Gross, Wolf. Sul calcolo della capacità elettrostatica di un conduttore. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **12**, 496–506 (1952).

Let S be the boundary of a given solid in space. The author designs an approximative method for the computation of the capacity C of S. Let $\varphi_0, \varphi_1, \varphi_2, \cdots$ be harmonic functions regular outside of S. The integral

$$I_n=\iint_S\left(1-\sum_{i=1}^n d_i\varphi_i\right)^2dS$$

is minimized by proper choice of the constants d_i and

$$C^{(n)}=-\frac{1}{4\pi}\sum_{i=1}^n d_i\iint\frac{\partial\varphi_i}{\partial n}dS$$

is considered as an approximation of C. The error $C-C^{(n)}$ is estimated in terms of I_n and the density σ of the unknown electrostatic equilibrium distribution. A particularly simple choice is $\varphi_1=1/r$ and φ_i, $i>1$, spherical harmonics without masses at infinity. The author applies this general method to the special case of a cube of edge 1 taking for the φ_i the spherical harmonics having the symmetry of the cube. Five approximations are worked out numerically and assuming an estimated value for $\iint\sigma^2dS$ the author claims that the approximation $C^{(5)}=0.6464$ yields C with an error not exceeding 0.032. *G. Szegő* (Stanford, Calif.).

Shen, D. W. C., and Broadbent, H. N. G. Analysis of partly symmetrical machines by means of unitary transformation. J. Franklin Inst. **254**, 473–485 (1952).

The theory of three-phase rotating machines is developed along stationary and rotating reference frames by using symmetrical-component reference frames and the associated unitary transformations. *G. Kron* (Schenectady, N. Y.).

Familier, H. Quelques aspects de la théorie de Bode. Ann. Radioélec. **5**, 36–53 (1950).

The general principles of Bode theory are summarized. The main mathematical relations are then applied to the investigation of transmission systems, and particularly to negative feed-back amplifiers. The results obtained are exemplified. *Author's summary.*

Quantum Mechanics

Epstein, Saul T. The causal interpretation of quantum mechanics. Physical Rev. (2) **89**, 319 (1953).

Bohm, David. Comments on a letter concerning the causal interpretation of the quantum theory. Physical Rev. (2) **89**, 319–320 (1953).

Fényes, I. Stochastischer Abhängigkeitscharakter der Heisenbergschen Ungenauigkeitsrelation. Naturwissenschaften **39**, 568 (1952).

de Broglie, Louis. La mécanique ondulatoire des systèmes de particules de même nature et la théorie de la double solution. C. R. Acad. Sci. Paris **235**, 1453–1455 (1952).

Morpurgo, G. Sulla corrispondenza tra elettrodinamica classica e quantistica. Nuovo Cimento (9) **9**, 808–817 (1952).

The classical Dirac-Eliezer equation for a point electron oscillator, $m\ddot{x}+\sigma x=(e^2/6\pi c^3)\dddot{x}$, is derived from the standard non-relativistic Hamiltonian of radiation theory. Using the same Hamiltonian, but considering the variables as q-numbers, the author shows that one obtains the same equation for the expectation values $\bar{x}$ of the position of the electron. Perturbation methods are not used in this derivation. The complete correspondence between classical and quantum electrodynamics in this case is considered as an indication that the non-physical run-away solutions of the Dirac-Eliezer equation should have their counterpart in quantum electrodynamics. *E. Gora* (Providence, R. I.).

Peierls, R. E. Commutation laws of relativistic field theory. Report of an International Conference on Elementary Particles, Bombay, 1950, pp. 11–14; discussion, pp. 15–16. The International Union of Pure and Applied Physics, Bombay, 1952.

A variational method of formulating the commutation laws of relativistic field theories is outlined. Replacement of the Lagrangian L by $L+\lambda\varphi_\alpha(x)\delta(x-x_0)$ modifies the solution $\varphi_\beta(x)$, to first order in λ, into

$$\varphi_\beta(x)+\lambda D^{\varphi_\beta(x)}_{\varphi_\alpha(x_0)}.$$

The arbitrariness involved in D is removed by prescribing boundary conditions, e.g., the condition

$$D^{\varphi_\beta(r,t)}_{\varphi_\alpha(r_0,t_0)}\to 0 \quad \text{as} \quad t\to-\infty$$

gives the retarded solution, while the condition

$$\mathfrak{A}^{\varphi_\beta(r,t)}_{\varphi_\alpha(r_0,t_0)}\to 0 \quad \text{as} \quad t\to+\infty$$

gives the advanced solution. It is then shown that any commutator can be written in the form

$$[A,B]=-i\operatorname{tr}\{D_A{}^B-\mathfrak{A}_A{}^B\}$$

where A, B are any two operator functions of the field quantities. It follows from the antisymmetry of $[A,B]$ that $D_A{}^B-\mathfrak{A}_A{}^B=-D_B{}^A+\mathfrak{A}_B{}^A$. If A and B are point functions, and B is taken at a later time than A, then $D_B{}^A$ and $\mathfrak{A}_A{}^B$ vanish and therefore $D_A{}^B=\mathfrak{A}_B{}^A$. This result is called the inversion theorem. In the discussion (Møller, Rosenfeld, Bhabha and others) the use of the inversion theorem and the conditions for its validity are considered. *E. Gora.*

Peierls, R. E. The commutation laws of relativistic field theory. Proc. Roy. Soc. London. Ser. A. **214**, 143–157 (1952).

A definition of Poisson brackets is given in terms of the Lagrangian of the system. This does not require the introduction of canonical variables. However, when such an introduction is possible, the new definition is shown to reduce to the conventional one with appropriate specialization. A manifestly covariant formulation of the customary commutation and anticommutation relations are readily obtained. A tentative discussion is also given of the extension to finding commutation relations for equations which cannot be put in canonical form. *K. M. Case.*

Caianiello, E. R., and Fubini, S. On the algorithm of Dirac spurs. Nuovo Cimento (9) **9**, 1218–1226 (1952).

In quantum electrodynamics and elsewhere it is often necessary to evaluate the spur S of products of Dirac matrices of the form $p^i\gamma_i+p^5$ where $i=1, 2, 3, 4$. The algorithm reduces the evaluation of S to that of some fourth-order determinants. It becomes especially efficient for products involving twelve or more factors, and increasingly so as the number of factors increases. A subsequent paper is promised in which all the contributions to a given process from the nth order Feynman-Dyson graphs will be "summed into one compact expression". *A. J. Coleman* (Toronto, Ont.).

Corben, H. C. The current density in quantum electrodynamics. Nuovo Cimento (9) **9**, 1071–1079 (1952).

The author has recently [Nuovo Cimento (9) **9**, 580–596 (1952); these Rev. **14**, 228] proposed a generalized quantum electrodynamics where the electric charge is represented by an operator. He shows now that one may choose two alternative expressions for the current density in this theory. One, j_i, has the same form as the usual current density of quantum electrodynamics; the other, j_i^*, differs from j_i by the divergence of a tensor so that the total charge inside a large volume is still the same with both expressions. One obtains, however, different expressions for the electromagnetic field, f_{ij} and $f^*{}_{ij}$, with j_i and j_i^*. The two current densities differ only in the presence of an electromagnetic field, the two fields only in the presence of a particle field. It is not necessary for the particles to carry a charge for f_{ij} and $f^*{}_{ij}$ to differ. Thus, also neutral particles can contribute to the field $f^*{}_{ij}$. *E. Gora* (Providence, R. I.).

Yennie, Donald R. Quantum corrections to classical non-linear meson theory. Physical Rev. (2) **88**, 527–536 (1952).

The interaction energy of a non-linear meson field with a given static source distribution is considered in the quantized theory. In addition to the classical result, which appears as the first approximation, and infinite renormalizations of the original parameters, there appear finite quantum corrections to the energy. Some of these are estimated and shown to increase with increasing source strength. It would then seem that in such a non-linear theory quantum corrections are not small perturbations and cannot be treated as such. *H. C. Corben* (Genoa).

Thermodynamics, Statistical Mechanics

Grad, Harold. Statistical mechanics of dynamical systems with integrals other than energy. J. Phys. Chem. **56**, 1039–1048 (1952).

The author develops the statistical mechanics of systems having a number $r>1$ of time-independent integrals which are smooth functions of the coordinates and momenta of the particles. In the conventional formulation, $r=1$ and the only integral is the energy. The author assumes r to be small compared to the number of degrees of freedom, as a conse-

quence of ergodicity. Applying the formalism of A. I. Khinchin [Mathematical foundations of statistical mechanics, Dover, New York, 1949; these Rev. **10**, 666], the theories of microcanonical, canonical, and grand canonical ensembles are developed, as well as a theory of "mechanical equilibrium" (generalizing the equilibrium at constant pressure). Except for the grand canonical ensemble, the entropy defined by the author differs from the conventional one by a term in $n \log n$, where n is the number of particles; it is therefore not additive. The formalism is illustrated by the example of a perfect gas of molecules with internal structure, the integrals considered being energy, momentum, and angular momentum.

The thermodynamics of irreversible processes in continuous media is then developed for systems with more than one integral, in the usual way, with a justification along the lines of Irving and Kirkwood [J. Chem. Phys. **18**, 817–829 (1950); these Rev. **12**, 230]. Consequences of conservation of angular momentum are briefly discussed. For certain proofs reference is made to the paper reviewed below.

L. Van Hove (Princeton, N. J.).

Grad, Harold. Statistical mechanics, thermodynamics, and fluid dynamics of systems with an arbitrary number of integrals. Comm. Pure Appl. Math. **5**, 455–494 (1952).

A detailed discussion is given of some of the problems treated in the paper reviewed above. They concern the canonical and microcanonical ensembles for systems with more than one integral. The example of a perfect gas is elaborated, as well as the presentation of the thermodynamics of continuous media. *L. Van Hove.*

Ikenberry, Ernest. The conservation of systems in phase space. Quart. Appl. Math. **9**, 195–203 (1951).

The author considers a system of N point molecules and the corresponding 6 N-dimensional phase space with equations of motion

$$\frac{dq_i}{dt}=\frac{\partial\epsilon}{\partial p_i}, \quad \frac{dp_i}{dt}=-\frac{\partial\epsilon}{\partial q_i}+F_i.$$

Following Gibbs, he considers a virtual ensemble (probability?) of $\mathbf{N}$ such systems (finite sample space of $\mathbf{N}$ members) represented by $\mathbf{N}$ points in the phase space and approximates this by a smooth density f. Elements of volume are $d\Omega_0$ in the full space and $d\Omega_1$ in the space complementary to the three physical space variables of molecule 1. The conservation of probability is

$$(*)\quad \frac{\partial f}{\partial t}+\sum\frac{\partial}{\partial q_i}\left(f\frac{dq_i}{dt}\right)+\sum\frac{\partial}{\partial p_i}\left(f\frac{dp_i}{dt}\right) =Df+f\sum\frac{\partial}{\partial q_i}\left(\frac{dq_i}{dt}\right)+f\sum\frac{\partial}{\partial p_i}\left(\frac{dp_i}{dt}\right)=0.$$

Writing $\psi=f\varphi(p_i, q_i, t)$, the author derives the equation

$$(**)\quad \frac{\partial}{\partial t}\int_{\Omega_0}\psi d\Omega_0=\int_{\Omega_0}\left[\frac{\partial\psi}{\partial t}+\frac{\partial}{\partial q_i}\left(\psi\frac{dq_i}{dt}\right)+\frac{\partial}{\partial p_i}\left(\psi\frac{dp_i}{dt}\right)\right]d\Omega_0 +\oint_{S_0}\psi[V_0(n)-v_0(n)]dS_0$$

for the rate of change of $\int\psi d\Omega_0$ in the domain Ω_0 bounded by the surface S_0 moving at velocity $V_0(v_0\sim dq_i/dt, dp_i/dt)$. For the average,

$$\langle\varphi\rangle=\int f\varphi d\Omega_1 \Big/ \int f d\Omega_1,$$

he obtains

$$(***)\quad \frac{\partial}{\partial t}\langle n\varphi\rangle+\frac{1}{m}\sum_{i=1}^{3}\frac{\partial}{\partial q_i}\langle np_i\varphi\rangle=n\langle D\varphi\rangle,$$

where n is the number density in physical space; from this he specializes ϕ to obtain the equations of conservation of mass, momentum, and energy.

The usual derivation of (***) is by direct integration of (*) over $d\Omega_1$ using the boundary conditions that f vanishes for large q_i and approaches zero rapidly for large p_i. The author objects to these boundary conditions and replaces them by the "boundary condition" that the surface integral in (**) vanishes. *H. Grad* (New York, N. Y.).

Cotter, J. R. Conduction of heat in a monatomic gas. Proc. Roy. Irish Acad. Sect. A. **55**, 1–28 (1952).

The author treats the Boltzmann equation for elastic spheres for the problem of a one-dimensional steady heat flow. The simplifications appropriate to steady flow and one space variable are introduced, and the Boltzmann equation is linearized corresponding to small values of the heat flow. The resulting integral equation is identical with the usual one (Fredholm type) occurring in the first Chapman-Enskog approximation. A new method of solution of this integral equation is presented in which the equation is first transformed into a second order ordinary differential equation and then integrated by an ingenious method yielding an alternating series and therefore simple error estimates.

H. Grad (New York, N. Y.).

Bernard, Jean-J. Sur une transformation du terme intégral de l'équation de Boltzmann. C. R. Acad. Sci. Paris **233**, 1348–1350 (1951).

The collision integral of the Boltzmann equation is frequently expressed in terms of the five integration variables $\mathbf{V}$ (the relative velocity of molecule s with respect to molecule 1 before collision) and two angles (θ, ϵ) locating the apse line with respect to $\mathbf{V}$ as pole. The author introduces a fixed coordinate system with respect to which the apse line has polar coordinates (η, φ) and $\mathbf{V}$ is replaced by the variables (x, y, z) through

$$\mathbf{V}=\{x+y,\ (x\tan\eta-y\cot\eta)\cos\varphi+z\sin\varphi,\ (x\tan\eta-y\cot\eta)\sin\varphi-z\cos\varphi\}.$$

This transformation yields a simple form of the Boltzmann equation for the case of a flow with one preferred direction (i.e., axial symmetry) for elastic sphere molecules.

H. Grad (New York, N. Y.).

Bernard, Jean-J. Probabilité de choc des molécules sphériques en fonction de leur vitesse d'agitation. C. R. Acad. Sci. Paris **234**, 510–512 (1952).

The author computes the frequency of collision of an elastic sphere molecule of given speed in a gas in equilibrium (Maxwellian velocity distribution) using the transformation of the above paper. The result (given in terms of Kummer functions) does not seem to differ from the usual one.

H. Grad (New York, N. Y.).

Marquet, Simone. Etude mathématique des équations de Boltzmann généralisées. C. R. Acad. Sci. Paris **234**, 2345–2347 (1952).

The author considers a relativistic Boltzmann equation [Marrot, J. Math. Pures Appl. (9) **25**, 93–159 (1946); these Rev. **8**, 187]

$$\frac{\partial F}{\partial t}+\sum u\frac{\partial F}{\partial x}+\sum X\frac{\partial F}{\partial \xi}=T(F)$$

where u and ξ represent the molecular velocity and momentum respectively (rest mass = 1). Exactly as for the classical Boltzmann equation, symmetry relations are obtained and the equations of hydrodynamics are derived. The stress tensor takes the form $\overline{u\xi}$ and is therefore not symmetric unless the mean velocity and mean momentum at a given point are parallel. *H. Grad.*

Grad, Harold. The profile of a steady plane shock wave. Comm. Pure Appl. Math. 5, 257–300 (1952).

This paper studies the structure of the steady one-dimensional shock wave from the point of view of kinetic theory, the analysis being based on the author's thirteen-moment approximation to the solution of the Boltzmann equation [same Comm. 2, 331–407 (1949); these Rev. 11, 473]. The mathematical problem which results is very similar to that based on the Navier-Stokes equations, and for sake of comparison the author develops the analytical details in parallel for both these equations and his thirteen-moment approximation. A fundamental difference between the two models is soon apparent: whereas the Navier-Stokes equations yield a shock wave for every shock strength, and even for arbitrary fluids, the thirteen-moment approximation provides shocks only up to pressure ratio 3.15 (Mach number 1.65), and is therefore applicable only to weak shocks. The author devotes the latter half of the paper to calculations of the shock profile and shock thickness for simple molecular models of monatomic gases, using an elegant computational method, the basic element of which is a series expansion in powers of the shock strength for a particular non-dimensional form of the flow quantities; the coefficients of the expansions can be computed once and for all for large classes of fluids, e.g., for all monatomic gases obeying the same viscosity law. The author's calculations of shock thickness show values up to 15% larger for the thirteen-moment approximation than for the Navier-Stokes equations, and in this respect they agree qualitatively with other kinetic theory treatments of the shock problem. However, it should be noted that all these comparisons with the continuum theory are based on kinetic theory values for the coefficients of viscosity and thermal conductivity in the Navier-Stokes equation rather than on the higher experimental values which broaden the shock front. In other respects the author's results and those of previous kinetic theory calculations [Zoller, Z. Physik 130, 1–38 (1951); these Rev. 13, 196; Mott-Smith, Physical Rev. (2) 82, 885–892 (1951); these Rev. 12, 891; Wang-Chang, Univ. of Michigan, Dept. of Eng. Rep. UMH-3-F(APL/JHU CM-503) (1948)] show very little agreement among themselves. The paper closes with an interesting observation that there is always a natural choice of reference mean free path in units of which the shock thickness is bounded away from zero however strong the shock and whatever the molecular model or form of viscosity dependence on temperature. *D. Gilbarg.*

Cox, R. T. Brownian motion in the theory of irreversible processes. Rev. Modern Physics 24, 312–320 (1952).

The author discusses Brownian motion problems (with no effort to attain mathematical rigor), including the motion of a particle in a uniform force field, and of an harmonic oscillator. The treatment is based in part on an earlier paper [Rev. Modern Physics 22, 238–248 (1950); these Rev. 12, 467]. *J. L. Doob* (Urbana, Ill.).

Okayama, Taisuke. Generalization of statistics. Progress Theoret. Physics 7, 517–534 (1952).

Bose and Fermi statistics are related to the identical and the alternating representation of the symmetrical group respectively. Other, intermediate statistics where a state may at most be occupied by n particles can be related to other representations of this group. Distribution functions for arbitrary n are derived from the generating function $g(xs^\nu)=1+xs^\nu+\cdots+x^n s^{n\nu}$. The wave functions used to describe such ensembles are defined by postulating that they should vanish identically when more than n particles occupy one and the same state. The equations for the time variation of the wave functions are then given in configuration space, and rewritten in the occupation number space. Finally, formulas for second quantization of the type

$$X_\alpha^n X_\alpha^* - nX_\alpha^* X_\alpha^n = nX_\alpha^{n-1},$$

where X_α denotes the annihilation operator and its conjugate X_α^* the creation operator, are established. The case $n=2$ is considered in detail. *E. Gora* (Providence, R. I.).

Martin, B., and ter Haar, D. Statistics of the three-dimensional ferromagnet. I. The variational method. Physica 18, 569–581 (1952).

In the second of their two papers introducing a matrix attack on the problem of order-disorder [Physical Rev. (2) 60, 263–276 (1941); these Rev. 3, 64], Kramers and Wannier described a process for referring the problem of crystal statistics to two problems of lower dimension, one for a single layer, the second for a double layer. The desired characteristic value is obtained by a maximizing process (cf. loc. cit., p. 268 et seq.). The authors have applied this process to three dimensions and generalized it so that on the basis of the same assumptions as those made in the Kramers and Wannier paper, the problem of evaluating λ is referred to the solution of a 4×4 and 2×2 matrix. The full results of their computational program are to be given in a later paper, but here λ is given for exp $(J/2kT)=1$, 1.05, 1.10, and 1.15. *F. J. Murray* (New York, N. Y.).

Yang, C. N. The spontaneous magnetization of a two-dimensional Ising model. Physical Rev. (2) 85, 808–816 (1952).

The Onsager-Kaufman solution of the two-dimensional Ising model problem is used to calculate the spontaneous magnetization. Below the critical temperature, the largest characteristic value is known to have two linearly independent characteristic vectors ψ_+ and ψ_-. The introduction of a small magnetic field removes this degeneracy and the characteristic vector for the largest characteristic value becomes $2^{-1/2}(\psi_+ + \psi_-)$. The spontaneous magnetization per atom is essentially the expected value of the total spin, and the matrix method of crystal statistics permits one to express this when the largest characteristic value and vector of the appropriate matrix is known. To evaluate the resulting expression, the author proceeds to make a skillful and very intricate computation using many of the Kaufman formulas, which, however, yields a rather simple result for the spontaneous magnetization. The problem is again the evaluation of the largest characteristic value of a product of simple matrices, and the essential device is the reduction of the $2^n\times 2^n$ matrices to $2n\times 2n$ matrices by the spinor relation. *F. J. Murray* (New York, N. Y.).